Noname manuscript No.
(will be inserted by the editor)

On the Normalized Laplacian Spectra of Random Geometric Graphs

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Received: date / Accepted: date

Abstract In this work, we study the spectrum of the normalized Laplacian and its regularized version for random geometric graphs (RGGs) in different scaling regimes. We consider n nodes distributed uniformly and independently on the d-dimensional torus $\mathbb{T}^d \equiv [0,1]^d$ and form an RGG by connecting two nodes when their ℓ_p-distance, $1 \leq p \leq \infty$, does not exceed a certain threshold r_n. Two scaling regimes for r_n are of special interest. One of these is the connectivity regime, in which the average vertex degree grows logarithmically in n. The second scaling regime is the thermodynamic regime, in which the average vertex degree is a constant. When d is fixed and $n \to \infty$, we prove that the limiting eigenvalue distribution (LED) of the normalized Laplacian matrix of RGGs converges to the Dirac distribution at one in the full range of the connectivity regime. In the thermodynamic regime, we propose an approximation for the LED and we provide a bound on the Levy distance between this approximation and the actual distribution. In particular, we show that the LED of the regularized normalized Laplacian matrix of an RGG can be approximated by the LED of the regularized normalized Laplacian of a deterministic geometric graph with nodes in a grid (DGG).

Keywords Random geometric graphs · Laplacian spectrum · Limiting eigenvalue distribution · Levy distance

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1 Introduction

The spectra of random matrices and random graphs have been extensively studied in the literature [1–5]. Spectral graph methods have become a fundamental tool in the analysis of large complex networks, and related disciplines, with a broad range of applications in telecommunication, machine learning, data mining, and web search. The first natural random graph model of complex networks is an Erdös-Rényi (ER) random graph [6] where edges between nodes appear with equal probabilities. This model has many appealing analytical properties but does not model important features of many real complex networks. In particular, the ER graph fails in describing clustering properties of graphs in which the geographical distance is a critical factor, as for example wireless ad-hoc network [7], sensor network [8], and the study of the dynamics of a viral spreading in a specific network of interactions [9], [10]. To properly model such networks, we consider a special class of graphs known as random geometric graphs (RGGs) [11]. Another very important motivation for the study of RGGs is their applications to statistics and learning. Many clustering techniques such as the nearest-neighbor technique in statistics and machine learning are based on the spatial structure of RGGs [12], [13].

Let us precisely define the RGG studied in this work. We consider a finite set $\mathcal{X}_n$ of n nodes, $x_1, ..., x_n$, distributed uniformly and independently on the d-dimensional torus $\mathbb{T}^d \equiv [0,1]^d$. Taking a torus $\mathbb{T}^d$ instead of a cube allows us not to consider boundary effects. Given a geographical distance $r_n > 0$, we form a graph by connecting two nodes $x_i, x_j \in \mathcal{X}_n$ if the ℓ_p-distance between them is at most r_n, i.e., $\|x_i - x_j\|_p \leq r_n$ with $p \in [1,\infty]$ (that is, either $p \in [1,\infty)$ or $p = \infty$), see Fig. 1(a). Here $\|.\|_p$ is the ℓ_p-metric on $\mathbb{R}^d$ defined as

$$\|x_i - x_j\|_p = \begin{cases} \left(\sum_{k=1}^{d} \mid x_i^{(k)} - x_j^{(k)} \mid^p\right)^{1/p} & \text{for } p \in [1,\infty), \\ \max\{\mid x_i^{(k)} - x_j^{(k)} \mid, \ 1 \leq k \leq d\} & \text{for } p = \infty, \end{cases}$$

where the case $p = 2$ gives the standard Euclidean metric on $\mathbb{R}^d$. When $p = \infty$, the maximum distance between two nodes is called the Chebyshev distance. Such graphs, denoted by $G(\mathcal{X}_n, r_n)$, are called RGGs and are extensively discussed in [11]. Typically, the function r_n is chosen such that $r_n \to 0$ when $n \to \infty$. Unlike ER graphs, the RGG is an inherently harder model to work with since the nature of the graph induces dependencies between edges.

The degree of a vertex in $G(\mathcal{X}_n, r_n)$ is the number of edges incident to it. Let $\theta^{(d)}$ denote the volume of the d-dimensional unit hypersphere in $\mathbb{T}^d$. Then, the average vertex degree in $G(\mathcal{X}_n, r_n)$ is equal to $a_n = \theta^{(d)} n r_n^d$ [11]. Often, RGGs present different properties depending on the average vertex degree a_n. Two different scaling regimes for a_n are of particular interest in this work. The first one is the connectivity regime, in which the average vertex degree a_n grows logarithmically in n or faster, i.e., $\Omega(\log(n))$[1] [11]. In real-world applications, networks may consist of several disconnected components, e.g., social interaction networks [14] and web graphs [15].

[1] The notation $f(n) = \Omega(g(n))$ indicates that $f(n)$ is bounded below by $g(n)$ asymptotically, i.e., $\exists K > 0$ and $n_o \in \mathbb{N}$ such that $\forall n > n_0 \ f(n) \geq K g(n)$.

In this case, the network structure falls in the thermodynamic regime. Therefore, the second scaling regime of interest is the thermodynamic regime, in which the average vertex degree is a constant γ, i.e., $a_n \equiv \gamma$ [11].

RGGs can be described by a variety of random matrices such as adjacency matrices, transition probability matrices and normalized Laplacian. The spectral properties of those random matrices are fundamental tools to predict and analyze complex networks behavior. The work in [16] investigates the combinatorial Laplacian spectra of RGGs and shows that the spectrum consists of both discrete and continuous parts. The discrete part of the spectrum is a collection of Dirac delta peaks at integer values. The work in [17] shows that the peaks appear mainly due to the existence of symmetric motifs[2] that occur abundantly in RGGs compared to ER graphs. Other works analyzed the symmetric motifs in RGGs and looked at the probabilities of their appearance [18].

Several works analyzed the spectra of Euclidean random matrices given by $H_n = f(\| x_i - x_j \|_2)$ when n and d grows large for different functions f [19], [20]. However, the obtained results cannot be applied to our problem as we assume that the dimension d stays fixed. The work in [20] also studies the LED of Euclidean random matrices H_n when the dimension d remains fixed. It shows under some conditions that in fact the LED of H_n converges to the Dirac distribution at 0. However, the results in [20] require the continuity of the function f and they cannot be applied to the step function considered in this work.

Regarding spectral properties of the adjacency matrix of RGGs, in [21] and [22], the authors show that the spectral distribution of the adjacency matrix has a limit in the thermodynamic regime as $n \to \infty$. Due to the difficulty to compute exactly this spectral measure, the work in [21] proposes an approximation for it as $\gamma \to \infty$. In [9], a closed form expression for the asymptotic spectral moments of the adjacency matrix of $G(\mathcal{X}_n, r_n)$ is derived in the connectivity regime. Then, an analytical upper bound for the spectral radius is derived in order to study the behavior of the viral infection in an RGG. Furthermore, the author in [23] shows that in the connectivity regime, the spectral measures of the transition probability matrix of the random walk in an RGG and in a deterministic geometric graph with nodes in a grid (DGG) converge to the same limit as $n \to \infty$. However, the author in [23] does not study the full range of the connectivity regime and in the proof of his result, a condition is enforced on the radius r_n using the concept of the minimum bottleneck matching distance. The condition enforced on r_n in [23] implies that for $\epsilon > 0$, the result holds only for RGGs with an average vertex degree a_n that scales as $\Omega\left(\log^{\epsilon}(n)\sqrt{n}\right)$, when $d = 1$, as $\Omega\left(\log^{\frac{3}{2}+\epsilon}(n)\right)$ when $d = 2$, as $\Omega\left(\log^{1+\epsilon}(n)\right)$ for $d \geq 3$.

Compared to [23], in this work we study the LED of the normalized Laplacian for RGGs in the connectivity regime for a wider range of scaling laws of the average vertex degree a_n, or equivalently a wider range of scaling laws of the radius r_n^d. More specifically, for $d \geq 1$, we show that the LEDs of the normalized Laplacian for RGGs and for DGGs converge to same limit when $a_n = \Omega(\log(n))$. Additionally, we extend the work in [23] and study the LED of the normalized Laplacian for RGGs formed by using any ℓ_p-metric, $p \in [1, \infty]$. Importantly, we study the LED of the normal-

[2] Symmetric motifs are subsets of nodes which have the same adjacencies.

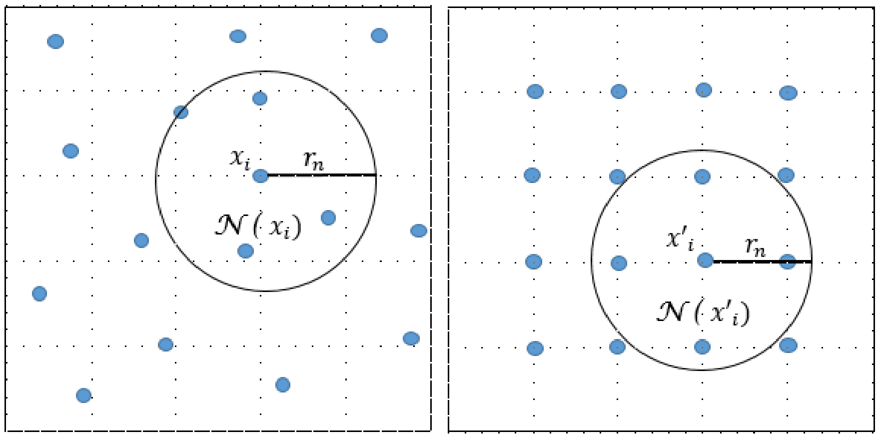
ized Laplacian for RGGs in the thermodynamic regime. To overcome the problem of singularities due to isolated nodes in the thermodynamic regime, we investigate the LED of the normalized Laplacian on a modified graph by adding auxiliary edges among all the nodes with a certain weight. The corresponding normalized Laplacian is known as the regularized normalized Laplacian [24].

To the best of our knowledge, explicit expressions for the LED of the combinatorial Laplacian and normalized Laplacian for RGGs are still not known in the full range of the scaling laws for the radius r_n^d in the connectivity regime, nor in the thermodynamic regime. In this work, we extend the work in [23] in various ways. Namely, we provide a bound on the Levy distance between the DGG-based approximation and the actual distribution. More precisely, in the connectivity regime, we prove that the LEDs of the normalized Laplacian for an RGG and for a DGG converge to the same limit in the full range of scaling law for a_n as $n \to \infty$. This result holds for any ℓ_p-metric and any fixed dimension $d \geq 1$. In addition, we show that when $a_n \geq \log^{1+\epsilon}(n)$ for $\epsilon > 0$, the rate of convergence is $\mathcal{O}\left(1/n^{(a_n/12\log(n))-1)}\right)$. In particular, we show that, when the average vertex degree $a_n = c\log(n)$, for $c > 24$, the rate of convergence is $\mathcal{O}\left(1/n^{c/12-1}\right)$, and when $c \leq 24$, a slower rate of convergence holds and scales as $\mathcal{O}\left(1/n\right)$. When the graph is dense, i.e., a_n scales as $\Omega(n)$, the LED of the normalized Laplacian for an RGG converges with the rate of convergence $\mathcal{O}\left(ne^{-n/12}\right)$. Finally, by using the Chebyshev distance, i.e., ℓ_∞-metric, we show that the LED of the normalized Laplacian of an RGG converges to the Dirac distribution at one in the full range of the connectivity regime as $n \to \infty$. In the thermodynamic regime, we show that the LED of the regularized normalized Laplacian of an RGG obtained by using any ℓ_p-metric can be approximated by the LED of the regularized normalized Laplacian of a DGG, with an error bound dependent upon the average vertex degree. Then, by using the Chebyshev distance, we provide an analytical approximation for the eigenvalues of the regularized normalized Laplacian for an RGG in the thermodynamic case.

The rest of this paper is organized as follows. In Section 2, we first describe the model, then we prove our main results on the concentration of the spectral measure of large RGGs. In particular, we provide an approximation for the eigenvalues of the regularized normalized Laplacian matrix in the thermodynamic regime, and we give its exact LED in the connectivity regime. Numerical results are given in Section 3 to validate the theoretical results by comparing the LED obtained analytically and by simulation. Finally, conclusions and implications are drawn in Section 4.

2 Spectral Analysis of Random Geometric Graphs

In this section, we study the spectrum of the regularized normalized Laplacian matrix of $G(\mathcal{X}_n, r_n)$ in the connectivity and thermodynamic regimes under any ℓ_p-metric as d remains fixed and $n \to \infty$. One important approach to study the spectrum of $G(\mathcal{X}_n, r_n)$ is based on analyzing the spectrum of a DGG [23]. As for RGGs, we let $\mathcal{D}_n$ be the set of n grid nodes that are at the intersections of all parallel hyperplanes with separation $n^{-1/d}$, and define a deterministic graph $G(\mathcal{D}_n, r_n)$ in the grid by connecting two nodes x'_i, $x'_j \in \mathcal{D}_n$ if $\parallel x'_i - x'_j \parallel_p \leq r_n$ for $p \in [1, \infty]$, see Fig.



(a) (b)

Fig. 1 Illustration of an RGG (a) and a DGG (b) for $n = 16$.

1(b). Given two nodes in $G(\mathcal{X}_n, r_n)$ or $G(\mathcal{D}_n, r_n)$, we assume that there is always at most one edge between them. There is no edge from a vertex to itself. Moreover, we assume that the edges are not directed. In the following, we define the normalized Laplacian matrix for $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{D}_n, r_n)$.

Let $\mathcal{N}(x_i)$ be the set of neighbors of vertex x_i in $G(\mathcal{X}_n, r_n)$ and $\mathcal{N}(x'_i)$ be the set of neighbors of vertex x'_i in $G(\mathcal{D}_n, r_n)$. Let $\mathcal{L}(\mathcal{X}_n)$ and $\mathcal{L}(\mathcal{D}_n)$ be the normalized Laplacian matrices for $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{D}_n, r_n)$, respectively, with entries,

$$\mathcal{L}(\mathcal{X}_n)_{ij} = \delta_{ij} - \frac{\chi[x_i \sim x_j]}{\sqrt{\mathbf{N}(x_i)\mathbf{N}(x_j)}}, \qquad \mathcal{L}(\mathcal{D}_n)_{ij} = \delta_{ij} - \frac{\chi[x'_i \sim x'_j]}{\sqrt{\mathbf{N}(x'_i)\mathbf{N}(x'_j)}}, \tag{1}$$

where $\mathbf{N}(x_i)$ and $\mathbf{N}(x'_i)$ are the sizes of the two sets $\mathcal{N}(x_i)$ and $\mathcal{N}(x'_i)$, respectively, and δ_{ij} is the Kronecker delta function. The term $\chi[x_i \sim x_j]$ takes unit value when there is an edge between nodes x_i and x_j in $G(\mathcal{X}_n, r_n)$ and zero otherwise, i.e.,

$$\chi[x_i \sim x_j] = \begin{cases} 1, \ \|x_i - x_j\|_p \leq r_n, \quad i \neq j \\ 0, \ \text{otherwise.} \end{cases}$$

A similar definition holds for $\chi[x'_i \sim x'_j]$ defined over the nodes in $G(\mathcal{D}_n, r_n)$. Recall that a_n denotes the average vertex degree in $G(\mathcal{X}_n, r_n)$ and, in particular, in the thermodynamic regime $a_n \equiv \gamma$ for a constant γ. In $G(\mathcal{D}_n, r_n)$, the number of neighbors of each vertex is the same. For simplicity, we denote this number by $a'_n = \mathbf{N}(x'_i)$. In particular, in the thermodynamic regime $\mathbf{N}(x'_i) = \gamma'$. We have also $\mathbf{N}(x_i, x'_i) = \sum_j \chi[x_i \sim x_j]\chi[x'_i \sim x'_j] \leq a'_n \ \forall \, i, j$.

Note that the above formal definition of the normalized Laplacian in (1) requires $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{D}_n, r_n)$ not to have isolated vertices. To overcome the problem of singularities due to isolated vertices in the termodynamic regime, we follow the scheme proposed in [24]. It corresponds to the normalized Laplacian matrix on a modified graph constructed by adding auxiliary edges among all the nodes with weight $\frac{\alpha}{n} > 0$. Specifically, the entries of the normalized Laplacian matrices are modified as

$$\hat{\mathcal{L}}(\mathcal{X}_n)_{ij} = \delta_{ij} - \frac{\chi[x_i \sim x_j] + \frac{\alpha}{n}}{\sqrt{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)}}, \quad \hat{\mathcal{L}}(\mathcal{D}_n)_{ij} = \delta_{ij} - \frac{\chi[x'_i \sim x'_j] + \frac{\alpha}{n}}{(a'_n + \alpha)}. \tag{2}$$

The corresponding matrices are referred to as the regularized normalized Laplacian matrices [24]. Observe that for $\alpha = 0$, (2) reduces to (1). The matrices $\hat{\mathcal{L}}(\mathcal{X}_n)$ and $\hat{\mathcal{L}}(\mathcal{D}_n)$ are symmetric, and consequently, their spectra consist of real eigenvalues. We denote by $\{\hat{\mu}_i, i = 1, .., n\}$ and $\{\hat{\lambda}_i, i = 1, .., n\}$ the sets of all real eigenvalues of the real symmetric square matrices $\hat{\mathcal{L}}(\mathcal{X}_n)$ and $\hat{\mathcal{L}}(\mathcal{D}_n)$ of order n, respectively. Then, the empirical spectral distribution functions of $\hat{\mathcal{L}}(\mathcal{X}_n)$ and $\hat{\mathcal{L}}(\mathcal{D}_n)$ are defined as

$$F^{\hat{\mathcal{L}}(\mathcal{X}_n)}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{I}\{\hat{\mu}_i \leq x\}, \quad \text{and} \quad F^{\hat{\mathcal{L}}(\mathcal{D}_n)}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{I}\{\hat{\lambda}_i \leq x\},$$

where $\mathrm{I}\{\mathrm{B}\}$ denotes the indicator of an event B. To show that $F^{\hat{\mathcal{L}}(\mathcal{D}_n)}$ is a good approximation for $F^{\hat{\mathcal{L}}(\mathcal{X}_n)}$ when n is large in both the connectivity and thermodynamic regimes, we use the Levy distance between the two distribution functions defined as follows.

Definition 1 ([25], page 257) (Levy Distance) Let F^A and F^B be two distribution functions on $\mathbb{R}$. The Levy distance $L(F^A, F^B)$ is defined as the infimum of all positive ϵ such that, for all $x \in \mathbb{R}$,

$$F^A(x - \epsilon) - \epsilon \leq F^B(x) \leq F^A(x + \epsilon) + \epsilon.$$

Lemma 1 ([2], page 614) *(Difference Inequality) Let A and B be two $n \times n$ Hermitian matrices with eigenvalues $\lambda_1, ..., \lambda_n$ and $\mu_1, ..., \mu_n$, respectively. Then,*

$$L^3(F^A, F^B) \leqslant \frac{1}{n} tr(A - B)^2,$$

where $L(F^A, F^B)$ denotes the Levy distance between the empirical distribution functions F^A and F^B of the eigenvalues of A and B, respectively.

In the following Lemma 2, we upper bound the Levy distance between the distribution functions $F^{\hat{\mathcal{L}}(\mathcal{X}_n)}$ and $F^{\hat{\mathcal{L}}(\mathcal{D}_n)}$ for any average vertex degree a_n.

Lemma 2 *(Upper Bound on the Levy Distance between $F^{\hat{\mathcal{L}}(\mathcal{X}_n)}$ and $F^{\hat{\mathcal{L}}(\mathcal{D}_n)}$) For $d \geqslant 1$ and $p \in [1, \infty]$, the Levy distance between the distribution functions $F^{\hat{\mathcal{L}}(\mathcal{X}_n)}$ and $F^{\hat{\mathcal{L}}(\mathcal{D}_n)}$ is upper bounded as follows:*

$$L^3\left(F^{\hat{\mathcal{L}}(\mathcal{X}_n)}, F^{\hat{\mathcal{L}}(\mathcal{D}_n)}\right) \leq \left| \frac{1}{n} \sum_i \sum_j \frac{\left(\chi[x_i \sim x_j] + \frac{\alpha}{n}\right)^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} - \frac{b}{n(a'_n+\alpha)^2} \right|$$
$$+ \left| \frac{2b}{n(a'_n+\alpha)^2} - \frac{2\left(\sum_i \mathbf{N}(x_i, x'_i) + \frac{\alpha}{n}\sum_i \mathbf{N}(x_i) + \alpha a'_n + \alpha^2\right)}{n(a'_n+\alpha)\left(\sum_i \sqrt{\mathbf{N}(x_i)+\alpha}\right)^2} \right|,$$

where $b = na'_n + \alpha^2 + 2\alpha a'_n$.

Proof See Appendix A.

We note that in the thermodynamic regime, for $d \geqslant 1$, the Levy distance between the distribution functions $F^{\hat{\mathcal{L}}(\mathcal{X}_n)}$ and $F^{\hat{\mathcal{L}}(\mathcal{D}_n)}$ is upper bounded as in Lemma 2 by letting $a_n \equiv \gamma$ being a positive constant. When the graph is connected, it is not necessary to work with the regularized normalized Laplacian, but we consider only the normalized Laplacian in (1) by enforcing $\alpha = 0$ in (2). Therefore, in the connectivity regime, the Levy distance between the distribution functions $F^{\mathcal{L}(\mathcal{X}_n)}$ and $F^{\mathcal{L}(\mathcal{D}_n)}$ is bounded as follows:

$$L^3\left(F^{\mathcal{L}(\mathcal{X}_n)}, F^{\mathcal{L}(\mathcal{D}_n)}\right) \leq \left| \frac{1}{n} \sum_i \sum_j \frac{\chi[x_i \sim x_j]^2}{\mathbf{N}(x_i)\mathbf{N}(x_j)} - \frac{1}{a'_n} \right|$$
$$+ \left| \frac{2}{a'_n} - \frac{2\sum_i \mathbf{N}(x_i, x'_i)}{na'_n\left(\sum_i \sqrt{\mathbf{N}(x_i)}\right)^2} \right|.$$

Next, we state a general theorem on the concentration of the regularized normalized Laplacian for any average vertex degree a_n. Then, we specify the result for both the connectivity and thermodynamic regimes.

Theorem 1 *(Concentration Theorem for the Regularized Normalized Laplacian Matrix in RGGs) For* $d \geq 1$, $p \in [1, \infty]$ *and* $t > \max\left[\frac{4(n+2\alpha)a'_n+4\alpha^2}{n(a'_n+\alpha)^2}, \frac{8(n+2\alpha)a_n+4\alpha^2}{n(a_n+\alpha)^2}\right]$, *the inequality holds:*

$$\begin{aligned} \mathrm{P}\left\{L^3\left(F^{\hat{\mathcal{L}}(\mathcal{X}_n)}, F^{\hat{\mathcal{L}}(\mathcal{D}_n)}\right) > t\right\} &\leq 2\mathrm{P}\left\{\left|\sum_i \mathbf{N}(x_i) - na_n\right| > \frac{tn(a_n+\alpha)}{16}\right\} \\ &+ 2\mathrm{P}\left\{\left|\sum_i \mathbf{N}(x_i) - na_n\right| > \frac{tn(a_n+\alpha)^2 - 4\alpha^2}{8\left(1+\frac{2\alpha}{n}\right)} - na_n\right\} \\ &+ \mathrm{P}\left\{\sum_i |a_n - \mathbf{N}(x_i)|^2 > \frac{tn(a_n+\alpha)^2}{8}\right\}. \end{aligned} \quad (3)$$

Proof See Appendix B.

Based on Theorem 1, we state the following corollary on the concentration of the regularized normalized Laplacian in RGGs specific to the thermodynamic regime.

Corollary 1 *(Concentration of the Regularized Normalized Laplacian in the Thermodynamic Regime) In the thermodynamic regime, i.e.,* $a_n \equiv \gamma$ *finite, for* $d \geq 1$, $p \in [1, \infty]$ *and* $t > \max\left[\frac{4(n+2\alpha)\gamma'+4\alpha^2}{n(\gamma'+\alpha)^2}, \frac{8(n+2\alpha)\gamma+4\alpha^2}{n(\gamma+\alpha)^2}\right]$*, we get*

$$\mathrm{P}\left\{L^3\left(F^{\hat{\mathcal{L}}(\mathcal{X}_n)}, F^{\hat{\mathcal{L}}(\mathcal{D}_n)}\right) > t\right\} \leq \frac{320(n-1)\vartheta}{tn^2(\gamma+\alpha)^2},$$

where $\vartheta = \left[\theta^{(d)} + 2(n-2)(\theta^{(d)})^2 r_n^d\right]$.

Under the conditions described above, for every $t > \max\left[\frac{4\gamma'}{(\gamma'+\alpha)^2}, \frac{8\gamma}{(\gamma+\alpha)^2}\right]$ *as* $n \to \infty$,

$$\lim_{n\to\infty} \mathrm{P}\left\{L^3\left(F^{\hat{\mathcal{L}}(\mathcal{X}_n)}, F^{\hat{\mathcal{L}}(\mathcal{D}_n)}\right) > t\right\} = 0.$$

Proof See Appendix C.

In the thermodynamic regime, $F^{\hat{\mathcal{L}}(\mathcal{D}_n)}$ approximates $F^{\hat{\mathcal{L}}(\mathcal{X}_n)}$ with an error bound of $\max\left[\frac{4}{\gamma'}, \frac{8}{\gamma}\right]$ when $n \to \infty$ and $\alpha \to 0$, which in particular implies that the error bound becomes small for large values of the degree.

In the following Lemma 3, we provide a lower bound on the degree of the vertices in $G(\mathcal{D}_n, r_n)$, useful for the following studies.

Lemma 3 *(Lower Bound on the Vertex Degree of the DGG) For any chosen* ℓ_p*-metric with* $p \in [1, \infty]$, $d \geq 1$ *and* $a_n \geq \frac{2d^{1+1/p}}{2d-1}$*, we have*

$$a_n' \geq \frac{a_n}{2d^{1+1/p}},$$

where a_n *is the average vertex degree of* $G(\mathcal{X}_n, r_n)$ *and* a_n' *is the degree of each vertex in* $G(\mathcal{D}_n, r_n)$.

Proof See Appendix D.

The following theorem shows that the LED of the normalized Laplacian for the RGG and the DGG converges to the same limit in the connectivity regime as $n \to \infty$.

Theorem 2 *(Concentration of the Normalized Laplacian in the Connectivity Regime) In the connectivity regime, i.e.,* $a_n = \Omega(\log(n))$*, for* $a_n \geq \frac{2d^{1+1/p}}{2d-1}$, $d \geq 1$, $p \in [1, \infty]$ *and* $t > \frac{8d^{1+1/p}}{a_n}$*, we have*

$$\mathrm{P}\left\{L^3\left(F^{\mathcal{L}(\mathcal{X}_n)}, F^{\mathcal{L}(\mathcal{D}_n)}\right) > t\right\} \leq \min\left[\frac{320(n-1)\vartheta}{tn^2(\gamma+\alpha)^2}, 6n\exp\left(-\frac{(a_n-r_n)}{12}\right)\right],$$

where $\vartheta = \left[\theta^{(d)} + 2(n-2)(\theta^{(d)})^2 r_n^d\right]$.

Under the conditions described above, for every $t > 0$,

$$\lim_{n\to\infty} \mathrm{P}\left\{L^3\left(F^{\mathcal{L}(\mathcal{X}_n)}, F^{\mathcal{L}(\mathcal{D}_n)}\right) > t\right\} = 0.$$

Proof See Appendix C.

We have $\vartheta < \theta^{(d)} a_n \left(\frac{1}{a_n} + 2\right)$ and $t > \frac{8d^{1+1/p}}{a_n}$ then,

$$A = \frac{320(n-1)\vartheta}{tn^2 a_n^2} < \frac{320\vartheta}{tna_n^2} < \frac{40\vartheta}{na_n d^{1+1/p}} = \frac{40\theta^{(d)}\left(\frac{1}{a_n}+2\right)}{nd^{1+1/p}},$$

and

$$B = \frac{6n}{\exp\left(\frac{a_n}{12}\left[1 - \frac{r_n}{a_n}\right]\right)} = \frac{6}{n^{\frac{a_n}{12\log(n)}\left[1-\frac{r_n}{a_n}\right]-1}},$$

therefore,

$$\mathrm{P}\left\{L^3\left(F^{\mathcal{L}(\mathcal{X}_n)}, F^{\mathcal{L}(\mathcal{D}_n)}\right) > t\right\} < \min\left[\frac{40\theta^{(d)}\left(\frac{1}{a_n}+2\right)}{nd^{1+1/p}}, \frac{6}{n^{\frac{a_n}{12\log(n)}\left[1-\frac{r_n}{a_n}\right]-1}}\right].$$

Note that, when $a_n = c\log(n)$, $c > 24$, then $B < A$ and the rate of convergence is $\mathcal{O}\left(1/n^{c/12-1}\right)$, and when $c \leq 24$, the rate of convergence is $\mathcal{O}(1/n)$. For $\epsilon > 0$ and $a_n \geq \log^{1+\epsilon}(n)$, the rate of convergence is $\mathcal{O}\left(1/n^{(a_n/12\log(n))-1}\right)$. When the graph is dense, i.e., a_n scales as $\Omega(n)$, the LED of the normalized Laplacian of the RGG converges to the Dirac measure at one with rate of convergence $\mathcal{O}\left(ne^{-n/12}\right)$. Hence, the result given in Theorem 2 shows that the LEDs of the normalized Laplacian for $G(\mathcal{X}_n, r_n)$ and $G(\mathcal{D}_n, r_n)$ converge to the same limit as $n \to \infty$ under any chosen ℓ_p-metric, and the convergence holds in the full range of the connectivity regime, i.e, $a_n = \Omega(\log(n))$.

In the following, we use the structure of the DGG to approximate the eigenvalues of the regularized normalized Laplacian matrix for $G(\mathcal{X}_n, r_n)$ in both the connectivity and thermodynamic regimes using the Chebyshev distance. Let us consider a d-dimensional DGG with $n = \mathrm{N}^d$ nodes and assume the use of the Chebyshev distance. Then, the degree of a vertex in $G(\mathcal{D}_n, r_n)$ is given as [16]

$$a'_n = (2k_n + 1)^d - 1, \quad \text{with } k_n = \lfloor \mathrm{N} r_n \rfloor,$$

where $\lfloor x \rfloor$ is the integer part, i.e., the greatest integer less than or equal to x. Note that when $d = 1$, the Chebyshev distance and the Euclidean distance are the same.

In the following Lemma 4, by using the expression of the eigenvalues of the adjacency matrix for a DGG under the Chebyshev distance [16], we approximate the eigenvalues of the regularized normalized Laplacian for $G(\mathcal{X}_n, r_n)$ when the number of nodes n is fixed and for any a_n. Then, we utilize this result to determine the LED of the normalized Laplacian in the connectivity regime as $n \to \infty$ in Corollary 2, and we provide the analytical approximation for the eigenvalues in the thermodynamic regime as n goes to infinity in Corollary 3.

Lemma 4 *(Eigenvalue Distribution of the Regularized Normalized Laplacian Matrix) When using the Chebyshev distance and* $d \geq 1$*, the eigenvalues of* $\hat{\mathcal{L}}(\mathcal{D}_n)$ *are given by*

$$\hat{\lambda}_{m_1,\ldots,m_d} = 1 - \frac{1}{(a'_n + \alpha)} \prod_{s=1}^{d} \frac{\sin(\frac{m_s\pi}{\mathrm{N}}(a'_n+1)^{1/d})}{\sin(\frac{m_s\pi}{\mathrm{N}})} + \frac{1 - \alpha\delta_{m_1,\ldots,m_d}}{(a'_n + \alpha)}, \qquad (4)$$

with $m_1, ..., m_d \in \{0, ...\mathrm{N}-1\}$ *and* $\delta_{m_1,...,m_d} = 1$ *when* $m_1, ..., m_d = 0$ *otherwise* $\delta_{m_1,...,m_d} = 0$*. In (4),* $n = \mathrm{N}^d$*,* $a'_n = (2k_n + 1)^d - 1$ *and* $k_n = \lfloor \mathrm{N} r_n \rfloor$*.*

Proof See Appendix E.

In the following Corollary 2, we provide the eigenvalue distribution of the normalized Laplacian matrix for $G(\mathcal{X}_n, r_n)$ in the connectivity regime as $a_n \geq 2d^{1+1/p}$ and $n \to \infty$.

Corollary 2 *(Eigenvalue Distribution of the Normalized Laplacian Matrix in the Connectivity Regime) In the connectivity regime, i.e.,* $a_n = \Omega(\log(n))$*, using the Chebyshev distance,* $a_n \geq 2d^{1+1/p}$*,* $d \geq 1$*, and letting* $\alpha \to 0$*, the eigenvalues of* $\mathcal{L}(\mathcal{D}_n)$ *are given by*

$$\lambda_{m_1,...,m_d} = 1 - \frac{1}{a'_n} \prod_{s=1}^{d} \frac{\sin(\frac{m_s \pi}{\mathrm{N}}(a'_n + 1)^{1/d})}{\sin(\frac{m_s \pi}{\mathrm{N}})} + \frac{1}{a'_n}, \tag{5}$$

with $m_1, ..., m_d \in \{0, ...\mathrm{N}-1\}$*. In (5),* $n = \mathrm{N}^d$*,* $a'_n = (2k_n+1)^d - 1$ *and* $k_n = \lfloor \mathrm{N} r_n \rfloor$*. Then, in particular, as* $n \to \infty$*, the LED of* $\mathcal{L}(\mathcal{D}_n)$ *converges to the Dirac measure at one.*

In the following Corollary 3, we approximate the eigenvalues of the regularized normalized Laplacian $\hat{\mathcal{L}}(\mathcal{X}_n)$ in the thermodynamic regime as $n \to \infty$.

Corollary 3 *(Eigenvalues of the Regularized Normalized Laplacian in the Thermodynamic Regime) In the thermodynamic regime, by using the Chebyshev distance,* $\gamma \geq 1$ *and* $d \geq 1$*, as* $n \to \infty$*, the eigenvalues of* $\hat{\mathcal{L}}(\mathcal{D}_n)$ *are given by*

$$\hat{\lambda}_{f_1,...,f_d} = 1 - \frac{1}{(\gamma' + \alpha)} \prod_{s=1}^{d} \frac{\sin(\pi f_s (\gamma' + 1)^{1/d})}{\sin(\pi f_s)} + \frac{1 - \alpha \delta_{f_1,...,f_d}}{(\gamma' + \alpha)}, \tag{6}$$

where $s \in \{1, ..., d\}$*,* $m_s \in \{0, ..., \mathrm{N}\}$ *and* $f_s = \frac{m_s}{\mathrm{N}}$ *in* $\mathbb{Q} \cap [0, 1]$ *with* $\mathbb{Q}$ *denotes the set of rational numbers. Also,* $\gamma' = (2\lfloor \gamma^{1/d} \rfloor + 1)^d - 1$ *and* $\delta_{f_1,...,f_d} = 1$ *when* $f_1, ..., f_d = 0$ *otherwise* $\delta_{f_1,...,f_d} = 0$*.*

In Lemma 4, we generalize the results given in [23], [20] and [21]. On one hand, in [23], the author shows that the spectral measures of the transition probability matrix of the random walk in RGGs and DGGs converge to the same limit in a specific range of the connectivity regime. On the other hand, in [20], the author shows that, for a fixed dimension d and $n \to \infty$, the LED of $H_n = f(\| x_i - x_j \|_2)$ converges to the Dirac measure in zero under some conditions on the function f. However, the techniques used in [20] cannot be applied to geometric graphs since the function f is required to be continuous. Additionally, the author of [21] characterizes the spectral measure of a normalized adjacency matrix in the dense regime. In contrast, in this work we study the LED of the normalized Laplacian matrix for an RGG formed by using any ℓ_p-metric, $1 \leq p \leq \infty$, and we show that it converges to the same limit as for the normalized Laplacian for a DGG in the full range of the connectivity regime as $n \to \infty$. In particular, we show that they converge to the Dirac measure at one as n goes to infinity in the full range of the connectivity regime.

3 Numerical Results

In this section, we validate our analytical results obtained in Section 2 by numerical computations.

More specifically, we corroborate our results on the spectrum of the regularized normalized Laplacian matrix of RGGs in the connectivity and thermodynamic regimes by comparing the simulated and the analytical spectra.

Fig. 2(a) illustrates the empirical spectral distribution in the thermodynamic regime of a realization for an RGG with $n = 30000$ vertices, $\alpha = 0.001$ and the corresponding DGG. The theoretical distribution is obtained from Corollary 3. We notice that the gap that appears between the eigenvalue distributions of the RGG and the DGG is upper bounded as in Corollary 1.

Here, we provide an additional example in the thermodynamic regime to quantify the error between $F^{\hat{\mathcal{L}}(\mathcal{X}_n)}$ and $F^{\hat{\mathcal{L}}(\mathcal{D}_n)}$ for different values of γ using the Chebyshev distance.

1) When $\gamma = 100$ and $\alpha = 10^{-3}$, $d = 1$ then as $n \to \infty$

$$P\left\{L^3\left(F^{\hat{\mathcal{L}}(\mathcal{X}_n)}, F^{\hat{\mathcal{L}}(\mathcal{D}_n)}\right) > \mathbf{0.019}\right\} \to 0.$$

2) When $\gamma = 120$ and $\alpha = 10^{-3}$, $d = 1$ then as $n \to \infty$

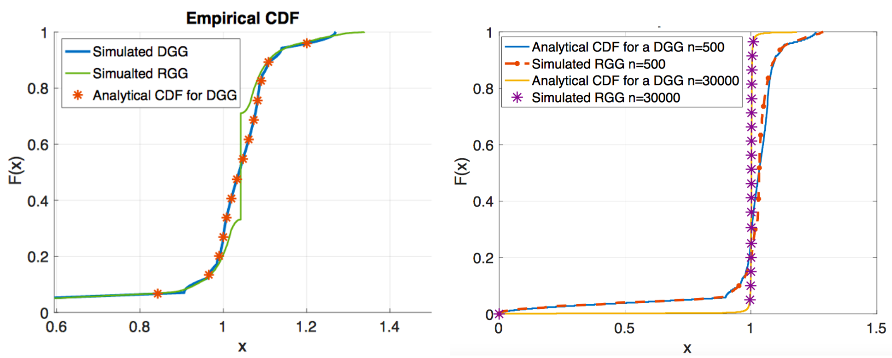
$$P\left\{L^3\left(F^{\hat{\mathcal{L}}(\mathcal{X}_n)}, F^{\hat{\mathcal{L}}(\mathcal{D}_n)}\right) > \mathbf{0.015}\right\} \to 0.$$

From these examples, we notice that for $\gamma = 100$, the LED of the regularized normalized Laplacian in the RGG can be approximated by the LED of a DGG with an error bound of 0.019 when $\alpha = 10^{-3}$. Then, as we increase the average vertex degree γ to $\gamma = 120$, we can notice a certain improvement. Therefore, the larger the average vertex degree γ is, the tighter the approximation becomes.

In Fig. 2(b) we compare the spectral distribution of a DGG (continuous lines) with the one for an RGG with increasing the number of nodes n (dashed line for $n = 500$ and star markers for $n = 30000$) in the connectivity regime. We notice that the curves corresponding to the RGG and the DGG match very well when n is large which confirm the concentration result given in Theorem 2. Also, it appears that by increasing n, the eigenvalue distribution converges to the Dirac measure at one, which confirms the result obtained in Corollary 2.

4 Conclusion

In this work, we studied in details the spectrum of RGGs in both the connectivity and thermodynamic regimes. In particular, we analyzed the LED of the regularized normalized Laplacian of RGGs. We first proposed an approximation for the LED and obtained a bound on the Levy distance between this approximation and the actual distribution. In the thermodynamic regime, where the average vertex degree is fixed, we found that the LEDs of the regularized normalized Laplacian matrix for an RGG can be approximated by the LED of a DGG. Then, we found that the LEDs of the



(a) Thermodynamic regime, $\gamma = 12$, $n = 30000$ and $\alpha = 0.001$.

(b) Connectivity regime for different n and $r_n = \log^{\frac{3}{2}}(n)/n$.

Fig. 2 Comparison between the simulated and the analytical spectral distributions of an RGG for $d = 1$.

normalized Laplacian for an RGG and for a DGG converge to the Dirac measure at one in the full range of the connectivity regime.

As future works, we will analyze the LED of the adjacency matrix in both the connectivity and thermodynamic regimes to derive their eigenvalue distributions. Furthermore, we intend to apply these theoretical results to test the hypothesis that a complex network has underlying geometric structure, or to analyze the spectral dimension of a network.

5 Acknowledgement

This research was funded by the French Government through the Investments for the Future Program with Reference: Labex UCN@Sophia-UDCBWN.

Appendices

A Proof of Lemma 2

In this Appendix, we upper bound the Levy distance between the distribution functions $F^{\hat{\mathcal{L}}(\mathcal{X}_n)}$ and $F^{\hat{\mathcal{L}}(\mathcal{D}_n)}$. The following lemma is useful for the following studies.

Lemma 5 *If $a_i \geq 0$ and $b_i > 0$ for all i, and there exists an $a_i > 0$, then*

$$\sum_{i=1}^{n} \frac{a_i}{b_i} > \frac{\sum\limits_{i=1}^{n} a_i}{\sum\limits_{i=1}^{n} b_i}.$$

In the following, we upper bound the Levy distance between $F^{\hat{\mathcal{L}}(\mathcal{X}_n)}$ and $F^{\hat{\mathcal{L}}(\mathcal{D}_n)}$.

$$
\begin{aligned}
L^3\left(F^{\hat{\mathcal{L}}(\mathcal{X}_n)}, F^{\hat{\mathcal{L}}(\mathcal{D}_n)}\right) &\le \frac{1}{n}\text{Trace}\left[\hat{\mathcal{L}}(\mathcal{X}_n) - \hat{\mathcal{L}}(\mathcal{D}_n)\right]^2 \\
&= \frac{1}{n}\sum_i\sum_j\left[\frac{\chi[x_i \sim x_j] + \frac{\alpha}{n}}{\sqrt{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)}} - \frac{\chi[x_i' \sim x_j'] + \frac{\alpha}{n}}{\sqrt{(a_n'+\alpha)(a_n'+\alpha)}}\right]^2 \\
&= \frac{1}{n}\sum_i\sum_j\left[\frac{\left(\chi[x_i \sim x_j] + \frac{\alpha}{n}\right)^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} + \frac{\left(\chi[x_i' \sim x_j'] + \frac{\alpha}{n}\right)^2}{(a_n'+\alpha)^2}\right] \\
&\qquad - \frac{2}{n}\sum_i\sum_j\frac{\left(\chi[x_i \sim x_j] + \frac{\alpha}{n}\right)\left(\chi[x_i' \sim x_j'] + \frac{\alpha}{n}\right)}{(a_n'+\alpha)\sqrt{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)}} \\
&\overset{(a)}{\le} \frac{b}{n(a_n'+\alpha)^2} - \frac{2\left(\sum\limits_i \mathbf{N}(x_i,x_i') + \frac{\alpha}{n}\sum\limits_i \mathbf{N}(x_i) + \alpha a_n' + \alpha^2\right)}{n(a_n'+\alpha)\left(\sum\limits_i\sqrt{\mathbf{N}(x_i)+\alpha}\right)^2} \\
&\qquad + \frac{1}{n}\sum_i\sum_j\frac{\left(\chi[x_i \sim x_j] + \frac{\alpha}{n}\right)^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} \\
&\overset{(b)}{\le} \left|\frac{1}{n}\sum_i\sum_j\frac{\left(\chi[x_i \sim x_j] + \frac{\alpha}{n}\right)^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} - \frac{b}{n(a_n'+\alpha)^2}\right| \\
&\qquad + \left|\frac{2b}{n(a_n'+\alpha)^2} - \frac{2\left(\sum\limits_i \mathbf{N}(x_i,x_i') + \frac{\alpha}{n}\sum\limits_i \mathbf{N}(x_i) + \alpha a_n' + \alpha^2\right)}{n(a_n'+\alpha)\left(\sum\limits_i\sqrt{\mathbf{N}(x_i)+\alpha}\right)^2}\right|,
\end{aligned}
$$

where, $b = na_n' + 2\alpha a_n' + \alpha^2$. Step (a) follows from $\sum\limits_j \chi[x_i \sim x_j] = \mathbf{N}(x_i), \sum\limits_j \chi[x_i' \sim x_j'] = a_n', \mathbf{N}(x_i,x_i') = \sum\limits_j \chi[x_i \sim x_j]\chi[x_i' \sim x_j']$ and Lemma 5. By applying the triangle inequality, step (b) follows. □

B Proof of Theorem 1

In this appendix, we provide an upper bound for the probability that the Levy distance between the distribution functions $F^{\hat{\mathcal{L}}(\mathcal{X}_n)}$ and $F^{\hat{\mathcal{L}}(\mathcal{D}_n)}$ is higher than $t > \max\left[\frac{4(n+2\alpha)a_n'+4\alpha^2}{n(a_n'+\alpha)^2}, \frac{8(n+2\alpha)a_n+4\alpha^2}{n(a_n+\alpha)^2}\right]$.

$$
\begin{aligned}
\mathrm{P}\left\{L^3\left(F^{\hat{\mathcal{L}}(\mathcal{X}_n)}, F^{\hat{\mathcal{L}}(\mathcal{D}_n)}\right) > t\right\} &\le \mathrm{P}\left\{\left|\frac{1}{n}\sum_i\sum_j\frac{\left(\chi[x_i \sim x_j] + \frac{\alpha}{n}\right)^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} - \frac{b}{n(a_n'+\alpha)^2}\right|\right. \\
&\qquad \left. + \left|\frac{2b}{n(a_n'+\alpha)^2} - \frac{2\left(\sum\limits_i \mathbf{N}(x_i,x_i') + \frac{\alpha}{n}\sum\limits_i \mathbf{N}(x_i) + \alpha a_n' + \alpha^2\right)}{n(a_n'+\alpha)\left(\sum\limits_i\sqrt{\mathbf{N}(x_i)+\alpha}\right)^2}\right| > t\right\} \\
&\le \mathrm{P}\left\{\left|\frac{1}{n}\sum_i\sum_j\frac{\left(\chi[x_i \sim x_j] + \frac{\alpha}{n}\right)^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} - \frac{b}{n(a_n'+\alpha)^2}\right| > \frac{t}{2}\right\} +
\end{aligned}
$$

$$\mathrm{P}\left\{\left|\frac{2b}{n(a_n'+\alpha)^2}-\frac{2\left(\sum_i \mathbf{N}(x_i,x_i')+\frac{\alpha}{n}\sum_i \mathbf{N}(x_i)+\alpha a_n'+\alpha^2\right)}{n(a_n'+\alpha)\left(\sum_i\sqrt{\mathbf{N}(x_i)+\alpha}\right)^2}\right|>\frac{t}{2}\right\}.$$

Define,

$$A=\left|\frac{2b}{n(a_n'+\alpha)^2}-\frac{2\left(\sum_i \mathbf{N}(x_i,x_i')+\frac{\alpha}{n}\sum_i \mathbf{N}(x_i)+\alpha a_n'+\alpha^2\right)}{n(a_n'+\alpha)\left(\sum_i\sqrt{\mathbf{N}(x_i)+\alpha}\right)^2}\right|,$$

and

$$B=\left|\frac{1}{n}\sum_i\sum_j\frac{\left(\chi[x_i\sim x_j]+\frac{\alpha}{n}\right)^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)}-\frac{b}{n(a_n'+\alpha)^2}\right|.$$

In the following, we upper bound $\mathrm{P}\left\{A>\frac{t}{2}\right\}$ and $\mathrm{P}\left\{B>\frac{t}{2}\right\}$.

First, we write $\mathrm{P}\left\{A>\frac{t}{2}\right\}$ as

$$\begin{aligned}
\mathrm{P}\left\{A>\frac{t}{2}\right\}&=\mathrm{P}\left\{\left|\frac{2b}{n(a_n'+\alpha)^2}-\frac{2\left(\sum_i \mathbf{N}(x_i,x_i')+\frac{\alpha}{n}\sum_i \mathbf{N}(x_i)+\alpha a_n'+\alpha^2\right)}{n(a_n'+\alpha)\left(\sum_i\sqrt{\mathbf{N}(x_i)+\alpha}\right)^2}\right|>\frac{t}{2}\right\}\\
&=\mathrm{P}\left\{\left|1-\frac{(a_n'+\alpha)\left[\sum_i \mathbf{N}(x_i,x_i')+\frac{\alpha}{n}\sum_i \mathbf{N}(x_i)+\alpha a_n'+\alpha^2\right]}{b\left(\sum_i\sqrt{\mathbf{N}(x_i)+\alpha}\right)^2}\right|>\frac{tn(a_n'+\alpha)^2}{4b}\right\}\\
&\overset{(a)}{=}\mathrm{P}\left\{b\left(\sum_i\sqrt{\mathbf{N}(x_i)+\alpha}\right)^2-(a_n'+\alpha)\left[\sum_i \mathbf{N}(x_i,x_i')+\frac{\alpha}{n}\sum_i \mathbf{N}(x_i)\right.\right.\\
&\qquad\left.\left.+\alpha a_n'+\alpha^2\right]>\frac{tn(a_n'+\alpha)^2}{4}\left(\sum_i\sqrt{\mathbf{N}(x_i)+\alpha}\right)^2\right\}\\
&=\mathrm{P}\left\{\left(b-\frac{tn(a_n'+\alpha)^2}{4}\right)\left(\sum_i\sqrt{\mathbf{N}(x_i)+\alpha}\right)^2\right.\\
&\qquad\left.>(a_n'+\alpha)\left[\sum_i \mathbf{N}(x_i,x_i')+\frac{\alpha}{n}\sum_i \mathbf{N}(x_i)+\alpha a_n'+\alpha^2\right]\right\}.
\end{aligned}$$

Note that $\sum_i \mathbf{N}(x_i,x_i')\le na_n'$ and $\mathbf{N}(x_i)\le n$. Then, in step (a) for n sufficiently large, we remove the absolute value because

$$\begin{aligned}
\frac{\sum_i \mathbf{N}(x_i,x_i')+\frac{\alpha}{n}\sum_i \mathbf{N}(x_i)+\alpha a_n'+\alpha^2}{b\left(\sum_i\sqrt{\mathbf{N}(x_i)+\alpha}\right)^2}&\le\frac{na_n'+\alpha n+\alpha a_n'+\alpha^2}{b\left(\sum_i\sqrt{\mathbf{N}(x_i)+\alpha}\right)^2}\\
&\le\frac{\left(1+\frac{\alpha}{a_n'}+\frac{\alpha}{n}+\frac{\alpha^2}{na_n'}\right)}{\left(1+\frac{2\alpha}{n}+\frac{\alpha^2}{na_n'}\right)n^2\alpha}\le 1.
\end{aligned}$$

Notice from the last equality that, $\frac{tn(a'_n+\alpha)^2}{4} > b \Leftrightarrow t > \frac{4na'_n + 4\alpha^2 + 8\alpha a'_n}{n(a'_n+\alpha)^2}$. Then

$$\mathrm{P}\left\{A > \frac{t}{2}\right\} = 0 \;\text{ for }\; t > \frac{4na'_n + 4\alpha^2 + 8\alpha a'_n}{n(a'_n+\alpha)^2}. \tag{7}$$

We continue further by bounding $\mathrm{P}\left\{B > \frac{t}{2}\right\}$ as

$$\begin{aligned}
\mathrm{P}\left\{B > \frac{t}{2}\right\} &= \mathrm{P}\left\{\left|\frac{1}{n}\sum_i\sum_j \frac{\left(\chi[x_i\sim x_j]+\frac{\alpha}{n}\right)^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} - \frac{b}{n(a'_n+\alpha)^2}\right| > \frac{t}{2}\right\}\\
&= \mathrm{P}\left\{\left|\frac{1}{n}\sum_i\sum_j \left(\frac{\left(\chi[x_i\sim x_j]+\frac{\alpha}{n}\right)^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} - \frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2}\right.\right.\right.\\
&\qquad\left.\left.\left. + \frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2}\right) - \frac{b}{n(a'_n+\alpha)^2}\right| > \frac{t}{2}\right\}\\
&\leqslant \mathrm{P}\left\{\left|\frac{1}{n}\sum_i\sum_j \left(\frac{\left(\chi[x_i\sim x_j]+\frac{\alpha}{n}\right)^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} - \frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2}\right)\right| > \frac{t}{4}\right\}\\
&\qquad + \mathrm{P}\left\{\left|\frac{1}{n}\sum_i\sum_j \frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2} - \frac{b}{n(a'_n+\alpha)^2}\right| > \frac{t}{4}\right\}.
\end{aligned}$$

Let

$$B_1 = \left|\frac{1}{n}\sum_i\sum_j \frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2} - \frac{b}{n(a'_n+\alpha)^2}\right|,$$

and

$$B_2 = \left|\frac{1}{n}\sum_i\sum_j \left(\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} - \frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2}\right)\right|.$$

In the following, we upper bound the two probabilities $\mathrm{P}\left\{B_1 > \frac{t}{4}\right\}$ and $\mathrm{P}\left\{B_2 > \frac{t}{4}\right\}$.

First, for $t > \frac{4na'_n+4\alpha^2+8\alpha a'_n}{n(a'_n+\alpha)^2}$, we write $\mathrm{P}\left\{B_1 > \frac{t}{4}\right\}$ as

$$\begin{aligned}
\mathrm{P}\left\{B_1 > \frac{t}{4}\right\} &= \mathrm{P}\left\{\left|\frac{1}{n}\sum_i\sum_j \frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2} - \frac{b}{n(a'_n+\alpha)^2}\right| > \frac{t}{4}\right\}\\
&= \mathrm{P}\left\{\left|\sum_i\sum_j \left(\chi[x_i\sim x_j]+\frac{\alpha}{n}\right)^2 - \frac{b(a_n+\alpha)^2}{(a'_n+\alpha)^2}\right| > \frac{nt(a_n+\alpha)^2}{4}\right\}\\
&\overset{(a)}{\leq} \mathrm{P}\left\{\left|\sum_i \mathbf{N}(x_i) - na_n\right| > \frac{nt(a_n+\alpha)^2 - 4\alpha^2}{4\left(1+\frac{2\alpha}{n}\right)} - na_n\right\}.
\end{aligned} \tag{8}$$

Step (a) follows from $\sum\limits_i\sum\limits_j \left(\chi[x_i\sim x_j]+\frac{\alpha}{n}\right)^2 = \left(1+\frac{2\alpha}{n}\right)\sum\limits_i \mathbf{N}(x_i) + \alpha^2$ and substituting the value of b.

We continue further by upper bounding the term $\mathrm{P}\left\{B_2 > \frac{t}{4}\right\}$ as

$$\begin{aligned}
\mathrm{P}\left\{B_2 > \frac{t}{4}\right\} &= \mathrm{P}\left\{\left|\sum_i\sum_j\left(\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} - \frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2}\right)\right| > \frac{tn}{4}\right\}\\
&= \mathrm{P}\left\{\sum_i\sum_j\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} - \sum_i\sum_j\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2} > \frac{tn}{4}\right\}\\
&\quad + \mathrm{P}\left\{\sum_i\sum_j\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2} - \sum_i\sum_j\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} > \frac{tn}{4}\right\}.
\end{aligned}$$

Define

$$\begin{aligned}
C_1 &= \sum_i\sum_j\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2} - \sum_i\sum_j\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)},\\
C_2 &= \sum_i\sum_j\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} - \sum_i\sum_j\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2}.
\end{aligned}$$

In the following, we upper bound $\mathrm{P}\left\{C_1 > \frac{tn}{4}\right\}$ and $\mathrm{P}\left\{C_2 > \frac{tn}{4}\right\}$.

We start first with $\mathrm{P}\left\{C_1 > \frac{tn}{4}\right\}$

$$\begin{aligned}
\mathrm{P}\left\{C_1 > \frac{tn}{4}\right\} &= \mathrm{P}\left\{\sum_i\sum_j\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2} - \sum_i\sum_j\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} > \frac{tn}{4}\right\}\\
&\overset{(a)}{\leq} \mathrm{P}\left\{\sum_i\sum_j\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2} - \frac{\left(1+\frac{2\alpha}{n}\right)\sum_i\mathbf{N}(x_i)+\alpha^2}{\left(\sum_i\mathbf{N}(x_i)+n\alpha\right)\sum_j(\mathbf{N}(x_j)+\alpha)} > \frac{tn}{4}\right\}\\
&= \mathrm{P}\left\{\sum_i\sum_j\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2} - \frac{\alpha\left[\left(\frac{1}{\alpha}+\frac{2}{n}\right)\sum_i\mathbf{N}(x_i)+\alpha\right]}{n\left(\frac{1}{n}\sum_i\mathbf{N}(x_i)+\alpha\right)\sum_j(\mathbf{N}(x_j)+\alpha)} > \frac{tn}{4}\right\}\\
&\overset{(b)}{\leq} \mathrm{P}\left\{\sum_{i,j}\frac{(\chi[x_i\sim x_j]+\frac{\alpha}{n})^2}{(a_n+\alpha)^2} - \frac{\alpha\left[\left(\frac{1}{\alpha}+\frac{2}{n}\right)\sum_i\mathbf{N}(x_i)+\alpha\right]}{n\left[\left(\frac{1}{\alpha}+\frac{2}{n}\right)\sum_i\mathbf{N}(x_i)+\alpha\right]\sum_j(\mathbf{N}(x_j)+\alpha)} > \frac{tn}{4}\right\}.
\end{aligned}$$

Step (a) follows from Lemma 5 and step (b) from $\left(\frac{1}{\alpha}+\frac{2}{n}\right)\sum_i\mathbf{N}(x_i)+\alpha > \frac{1}{n}\sum_i\mathbf{N}(x_i)+\alpha$.

$$\begin{aligned}
\mathrm{P}\left\{C_1 > \frac{tn}{4}\right\} &= \mathrm{P}\left\{(n+2\alpha)\sum_i\mathbf{N}(x_i) - \frac{\alpha(a_n+\alpha)^2}{\sum_j\mathbf{N}(x_j)+n\alpha} > \frac{tn^2(a_n+\alpha)^2}{4} - n\alpha^2\right\}\\
&\leq \mathrm{P}\left\{(n+2\alpha)\left|\sum_i\mathbf{N}(x_i)-na_n\right| + \left|(n+2\alpha)\,na_n - \frac{\alpha(a_n+\alpha)^2}{\sum_j\mathbf{N}(x_j)+n\alpha}\right|\right.\\
&\qquad\qquad\left. > \frac{tn^2(a_n+\alpha)^2}{4} - n\alpha^2\right\}.
\end{aligned}$$

Then,

$$\mathrm{P}\left\{C_1 > \frac{tn}{4}\right\} \leq \mathrm{P}\left\{\left|(n+2\alpha)\,na_n - \frac{\alpha(a_n+\alpha)^2}{\sum_j \mathbf{N}(x_j) + n\alpha}\right| > \frac{tn^2(a_n+\alpha)^2 - 4n\alpha^2}{8}\right\}$$
$$+\,\mathrm{P}\left\{\left|\sum_i \mathbf{N}(x_i) - na_n\right| > \frac{tn(a_n+\alpha)^2 - 4\alpha^2}{8\left(1+\frac{2\alpha}{n}\right)}\right\}.$$

Let

$$C_3 = \mathrm{P}\left\{\left|\sum_i \mathbf{N}(x_i) - na_n\right| > \frac{tn(a_n+\alpha)^2 - 4\alpha^2}{8\left(1+\frac{2\alpha}{n}\right)}\right\}, \tag{9}$$

and

$$D = \left|(n+2\alpha)\,na_n - \frac{\alpha(a_n+\alpha)^2}{\sum_j \mathbf{N}(x_j) + n\alpha}\right|.$$

$$\mathrm{P}\left\{D > \frac{tn^2(a_n+\alpha)^2 - 4n\alpha^2}{8}\right\} = \mathrm{P}\left\{\left|(n+2\alpha)\,na_n - \frac{\alpha(a_n+\alpha)^2}{\sum_j \mathbf{N}(x_j) + n\alpha}\right| > \frac{tn^2(a_n+\alpha)^2 - 4n\alpha^2}{8}\right\}$$
$$\stackrel{(a)}{\leq} \mathrm{P}\left\{\left[(n+2\alpha)\,na_n + \frac{4n\alpha^2 - tn^2(a_n+\alpha)^2}{8}\right] \times \left(\sum_j \mathbf{N}(x_j) + n\alpha\right) > \alpha(a_n+\alpha)^2\right\}.$$

Step (a) follows from the inequality $(n+2\alpha)\,na_n > \frac{\alpha(a_n+\alpha)^2}{\sum_j \mathbf{N}(x_j) + n\alpha}$. Notice that $8\,(n+2\alpha)\,na_n + 4n\alpha^2 < tn^2(a_n+\alpha)^2 \Leftrightarrow t > \frac{8\,(n+2\alpha)\,a_n + 4\alpha^2}{n(a_n+\alpha)^2}$. Then,

$$\mathrm{P}\left\{D > \frac{tn^2(a_n+\alpha)^2 - 4n\alpha^2}{8}\right\} = 0 \;\text{ for }\; t > \frac{8\,(n+2\alpha)\,a_n + 4\alpha^2}{n(a_n+\alpha)^2}. \tag{10}$$

Finally, we upper bound the remaining probability $\mathrm{P}\left\{C_2 > \frac{tn}{4}\right\}$ as

$$\mathrm{P}\left\{C_2 > \frac{tn}{4}\right\} = \mathrm{P}\left\{\sum_i\sum_j \frac{(\chi[x_i\sim x_j] + \frac{\alpha}{n})^2}{(\mathbf{N}(x_i)+\alpha)(\mathbf{N}(x_j)+\alpha)} - \sum_i\sum_j \frac{(\chi[x_i\sim x_j] + \frac{\alpha}{n})^2}{(a_n+\alpha)^2} > \frac{tn}{4}\right\}$$
$$= \mathrm{P}\left\{\sum_i\sum_j \frac{(\chi[x_i\sim x_j] + \frac{\alpha}{n})}{(\mathbf{N}(x_i)+\alpha)}\frac{(\chi[x_i\sim x_j] + \frac{\alpha}{n})}{(\mathbf{N}(x_j)+\alpha)} - \sum_i\sum_j \frac{(\chi[x_i\sim x_j] + \frac{\alpha}{n})^2}{(a_n+\alpha)^2} > \frac{tn}{4}\right\}$$
$$\stackrel{(a)}{\leq} \mathrm{P}\left\{\left(\sum_i\sum_j \frac{(\chi[x_i\sim x_j] + \frac{\alpha}{n})^2}{(\mathbf{N}(x_i)+\alpha)^2}\right)^{\frac{1}{2}}\left(\sum_i\sum_j \frac{(\chi[x_i\sim x_j] + \frac{\alpha}{n})^2}{(\mathbf{N}(x_j)+\alpha)^2}\right)^{\frac{1}{2}} - \sum_i\sum_j \frac{(\chi[x_i\sim x_j] + \frac{\alpha}{n})^2}{(a_n+\alpha)^2} > \frac{tn}{4}\right\}.$$

$$\mathrm{P}\left\{C_2 > \frac{tn}{4}\right\} = \mathrm{P}\left\{\sum_i\sum_j \frac{\left(\chi[x_i \sim x_j] + \frac{\alpha}{n}\right)^2}{(\mathbf{N}(x_i)+\alpha)^2} - \sum_i\sum_j \frac{(\chi[x_i \sim x_j] + \frac{\alpha}{n})^2}{(a_n+\alpha)^2} > \frac{tn}{4}\right\}. \tag{11}$$

Step (a) follows from applying Cauchy-Schwarz inequality. Then

$$\begin{aligned}
\mathrm{P}\left\{C_2 > \frac{tn}{4}\right\} &\leq \mathrm{P}\left\{\sum_i\sum_j \left(\chi[x_i \sim x_j] + \frac{\alpha}{n}\right)^2 \left|\frac{1}{(\mathbf{N}(x_i)+\alpha)^2} - \frac{1}{(a_n+\alpha)^2}\right| > \frac{tn}{4}\right\}\\
&\leq \mathrm{P}\left\{\sum_i \left|(a_n+\alpha)^2 - (\mathbf{N}(x_i)+\alpha)^2\right| > \frac{tn(a_n+\alpha)^2}{4}\right\}\\
&\leq \mathrm{P}\left\{\sum_i |a_n - \mathbf{N}(x_i)|^2 + 2(a_n+\alpha)\sum_i |a_n - \mathbf{N}(x_i)| > \frac{tn(a_n+\alpha)^2}{4}\right\}\\
&\leq \mathrm{P}\left\{\sum_i |a_n - \mathbf{N}(x_i)|^2 > \frac{tn(a_n+\alpha)^2}{8}\right\} + 2\mathrm{P}\left\{\left|\sum_i \mathbf{N}(x_i) - na_n\right| > \frac{tn(a_n+\alpha)}{16}\right\}.
\end{aligned} \tag{12}$$

Finally, $\mathrm{P}\left\{C_1 > \frac{tn}{4}\right\}$ is upper bounded by the sum of (9) and (10). We Use this result combined with the upper bound of $\mathrm{P}\left\{C_2 > \frac{tn}{4}\right\}$ given in (12) to upper bound the term $\mathrm{P}\left\{B_2 > \frac{t}{4}\right\}$. Then, apply the new upper bound with (7) and (8) to upper bound (3) and therefore Theorem 1 follows. □

C Proof of Corollary 1 and Theorem 2

In this Appendix, we show that the LED of the regularized normalized Laplacian for a DGG is a good approximation for the LED of the regularized normalized Laplacian for an RGG in both the connectivity and thermodynamic regimes.

To upper bound the terms obtained in Theorem 1, we use the Chebyshev inequality. Notice that $\sum\limits_i \mathbf{N}(x_i)/2$ that appears in Theorem 1 counts the number of edges in $G(\mathcal{X}_n, r_n)$. For convenience, we denote $\sum\limits_i \mathbf{N}(x_i)/2$ as ξ_n. In order to apply the Chebyshev inequality, we determine the variance of the number of edges, i.e.,$\mathrm{Var}(\xi_n)$ in the following lemma.

Lemma 6 *(Variance of ξ_n) When $x_1, ..., x_n$ are i.i.d. uniformly distributed in the d-dimensional unit torus $\mathbb{T}^d = [0,1]$*

$$\mathrm{Var}\left(\xi_n\right) \leq \frac{(n-1)}{n}\left[\theta^{(d)} + 2(n-2)(\theta^{(d)})^2 r_n^d\right].$$

Proof The proof follows along the same lines of Proposition A.1 in [26] when extended to a unit torus and applied to i.i.d. and uniformly distributed nodes.

Let $\vartheta = \left[\theta^{(d)} + 2(n-2)(\theta^{(d)})^2 r_n^d\right]$. In the following, we upper bound the probabilities given in Theorem 1 using Lemma 6 and the Chebyshev inequality. We start by upper bounding the first term as

$$\begin{aligned}
&\mathrm{P}\left\{\left|\sum_i \mathbf{N}(x_i) - na_n\right| > \frac{tn(a_n+\alpha)^2 - 4\alpha^2}{4\left(1+\frac{2\alpha}{n}\right)} - na_n\right\}\\
&= \mathrm{P}\left\{|\xi_n - \mathbb{E}\xi_n| > \frac{tn(a_n+\alpha)^2 - 4\alpha^2}{8\left(1+\frac{2\alpha}{n}\right)} - na_n\right\}\\
&\leq \frac{8^2\left(1+\frac{2\alpha}{n}\right)^2 \mathrm{Var}(\xi_n)}{\left[tn(a_n+\alpha)^2 - 4\alpha^2 - 8(1+\frac{2\alpha}{n})na_n\right]^2}\\
&= \frac{8^2\left(1+\frac{2\alpha}{n}\right)^2 (n-1)\vartheta}{n\left[tn(a_n+\alpha)^2 - 4\alpha^2 - 8(1+\frac{2\alpha}{n})na_n\right]^2},
\end{aligned}$$

and

$$\mathrm{P}\left\{\left|\sum_i \mathbf{N}(x_i) - na_n\right| > \frac{tn(a_n+\alpha)}{16}\right\} \leq \frac{32^2 \mathrm{Var}(\xi_n)}{t^2n^2(a_n+\alpha)^2}$$
$$= \frac{32^2(n-1)\vartheta}{t^2n^3(a_n+\alpha)^2}. \quad (13)$$

Finally, we upper bound the last term as

$$\mathrm{P}\left\{\sum_i |\mathbf{N}(x_i) - a_n|^2 > \frac{nt(a_n+\alpha)^2}{8}\right\} \leq \mathrm{P}\left\{\left(\sum_i |\mathbf{N}(x_i) - a_n|\right)^2 > \frac{nt(a_n+\alpha)^2}{8}\right\}$$
$$= \mathrm{P}\left\{\sum_i |\mathbf{N}(x_i) - a_n| > \frac{(a_n+\alpha)\sqrt{nt}}{2\sqrt{2}}\right\}.$$

Then,

$$\mathrm{P}\left\{\sum_i |\mathbf{N}(x_i) - a_n|^2 > \frac{nt(a_n+\alpha)^2}{8}\right\} \leq 2\mathrm{P}\left\{\left|\sum_i \mathbf{N}(x_i) - na_n\right| > \frac{(a_n+\alpha)\sqrt{nt}}{2\sqrt{2}}\right\}$$
$$= 2\mathrm{P}\left\{|\xi_n - \mathbb{E}\xi_n| > \frac{(a_n+\alpha)\sqrt{nt}}{4\sqrt{2}}\right\}$$
$$\leq \frac{64(n-1)\vartheta}{tn^2(a_n+\alpha)^2}. \quad (14)$$

Corollary 1 for the thermodynamic regime is obtained from upper bounding the terms in the r.h.s of (3) in Theorem 1 obtained in (C), (13) and (14). In addition, by letting $\alpha \to 0$, the provided upper bounds hold in the connectivity case.

In the following, we propose a tighter upper bound for the Levy distance between $F^{\mathcal{L}(\mathcal{D}_n)}$ and $F^{\mathcal{L}(\mathcal{X}_n)}$ in the connectivity regime. More precisely, the following upper bounds are tighter when the average vertex degree scales as $\Omega(\log^{1+\epsilon}(n))$ or for $a_n = c\log(n)$ when $c > 24$.

First, observe that the number of nodes that fall in an arbitrary interval of radius r_n follows a binomial distribution. Then in order to derive the distribution of $\mathbf{N}(x_i)$, we need to derive the distribution of the nodes that fall in a ball centered in x_i. To derive the distribution of $\mathbf{N}(x_i)$ in the connectivity regime, we throw at random a node which will be in a random position and we are left with $n-1$ nodes. Then, we take a ball of size $2r_n$, centered around the thrown node. If we throw randomly the remaining $n-1$ nodes, then, $\mathbf{N}(x_i)$ will be a random variable binomially distributed with parameters $(n-1, \theta^{(d)}r_n)$, i.e.,

$$\mathrm{P}(\mathbf{N}(x_i) = k) = \binom{n-1}{k}(\theta^{(d)}r_n)^k(1-\theta^{(d)}r_n)^{n-k-1}, \text{ for } k = 0, ..., n-1.$$

To upper bound the terms in the r.h.s of (3) given in Theorem 1, we introduce upper bounds for a binomially distributed random variable X appropriate for large deviations. These results play a key role to establish the relation between $F^{\mathcal{L}(\mathcal{X}_n)}$ and $F^{\mathcal{L}(\mathcal{D}_n)}$.

Lemma 7 *([27], Theorem 2.1, page 26) Let $X \sim Bin(n,p)$, $\mathbb{E}X = np$, then*

$$\mathrm{P}\{X \geq \mathbb{E}X + t\} \leq \exp\left(-\frac{t^2}{2(\mathbb{E}X + t/3)}\right), \qquad t \geq 0.$$

Lemma 8 *([27], corollary 2.3, page 27) Let $X \sim Bin(n,p)$, $\mathbb{E}X = np$ and $0 < t \leq 3/2$, then*

$$\mathrm{P}\{|X - \mathbb{E}X| \geq t\mathbb{E}X\} \leq 2\exp\left(-\frac{t^2}{3}\mathbb{E}X\right).$$

We upper bound the first term in the r.h.s of (3) in Theorem 1 by using Lemmas 7 and 8 as follows:

$$\begin{aligned}
\mathrm{P}\left\{\left|\sum_i \mathbf{N}(x_i)-na_n\right|>\frac{tna_n^2}{4}-na_n\right\} &\overset{(a)}{\leq} \mathrm{P}\left\{\sum_i |\mathbf{N}(x_i)-a_n|>\frac{nta_n^2}{4}-na_n\right\}\\
&\overset{(b)}{\leq} n\mathrm{P}\left\{|\mathbf{N}(x_i)-a_n|>\frac{ta_n^2}{4}-a_n\right\}\\
&\leq n\mathrm{P}\left\{\mathbf{N}(x_i)>\frac{ta_n^2}{4}\right\}+n\mathrm{P}\left\{\mathbf{N}(x_i)<2a_n-\frac{ta_n^2}{4}\right\}\\
&\leq n\exp\left(-\frac{(ta_n-4)^2}{16\left(\frac{t}{6}-\frac{2r_n}{a_n^2}+\frac{4}{3a_n}\right)}\right), \quad \text{for } t>\frac{8}{a_n}.
\end{aligned} \tag{15}$$

Step (a) follows from $\left|\sum_i z_i\right| \leq \sum_i |z_i|$ and step (b) from $\sum_i |\mathbf{N}(x_i)-a_n| \leq n\,|\mathbf{N}(x_i)-a_n|$.

Now, instead of upper bounding the two last probabilities given in Theorem 1, we go back to Appendix B and upper bound (11) by letting $\alpha \to 0$.

$$\begin{aligned}
\mathrm{P}\left\{C_2>\frac{tn}{4}\right\} &\leq \mathrm{P}\left\{\sum_i\sum_j \frac{\chi[x_i\sim x_j]^2}{\mathbf{N}(x_i)^2}-\sum_i\sum_j\frac{\chi[x_i\sim x_j]^2}{a_n^2}>\frac{tn}{4}\right\}\\
&\leq n\mathrm{P}\left\{\frac{1}{\mathbf{N}(x_i)}-\frac{\mathbf{N}(x_i)}{a_n^2}>\frac{t}{4}\right\}\\
&= n\mathrm{P}\left\{a_n^2-\mathbf{N}(x_i)^2>\frac{ta_n^2\mathbf{N}(x_i)}{4}\right\}
\end{aligned}$$

$$\begin{aligned}
\mathrm{P}\left\{C_2>\frac{tn}{4}\right\} &= n\mathrm{P}\left\{\left[a_n^2-\mathbf{N}(x_i)^2\right]+\frac{ta_n^2}{4}\left[a_n-\mathbf{N}(x_i)\right]>\frac{ta_n^3}{4}\right\}\\
&\leq n\mathrm{P}\left\{\left|a_n^2-\mathbf{N}(x_i)^2\right|>\frac{ta_n^3}{8}\right\}+n\mathrm{P}\left\{|a_n-\mathbf{N}(x_i)|>\frac{a_n}{2}\right\}\\
&\leq n\mathrm{P}\left\{-\mathbf{N}(x_i)^2>\frac{ta_n^3}{8}-a_n^2\right\}+n\mathrm{P}\left\{|a_n-r_n-\mathbf{N}(x_i)|>\frac{a_n-r_n}{2}\right\}\\
&\quad + n\mathrm{P}\left\{\mathbf{N}(x_i)>a_n-r_n+\sqrt{\frac{ta_n^3}{8}+a_n^2}-a_n+r_n\right\}\\
&= n\mathrm{P}\left\{-\mathbf{N}(x_i)^2>\frac{ta_n^3}{8}-a_n^2\right\}+n\mathrm{P}\left\{|\mathbf{N}(x_i)-\mathbb{E}\mathbf{N}(x_i)|>\frac{a_n-r_n}{2}\right\}\\
&\quad + n\mathrm{P}\left\{\mathbf{N}(x_i)>\mathbb{E}\mathbf{N}(x_i)+\sqrt{\frac{ta_n^3}{8}+a_n^2}-a_n+r_n\right\}.
\end{aligned}$$

Then, applying Lemma 7 and 8 and for $t>\frac{8}{a_n}$, yields

$$\mathrm{P}\left\{C_2>\frac{tn}{4}\right\} \leq 2n\exp\left(-\frac{(a_n-r_n)}{12}\right)+n\exp\left(\frac{-3a_n\left[\sqrt{\frac{t}{8}a_n+1}-1+\frac{r_n}{a_n}\right]^2}{2\left[2-\frac{2r_n}{a_n}+\sqrt{\frac{t}{8}a_n+1}\right]}\right). \tag{16}$$

Finally, taking the upper bounds found by using the Chebyshev inequality in $(C), (13)$ and (14) combined with the upper bounds found by using Lemmas 7 and 8, i.e., $(15), (16)$ all together, then by applying Lemma 3 and letting $\alpha \to 0$, Theorem 2 follows. □

D Proof of Lemma 3

In this appendix we show that for $a_n \geq \dfrac{2d^{1+1/p}}{2d-1}$ and any ℓ_p-metric, $p \in [1,\infty]$, the vertex degree a'_n of $G(\mathcal{D}_n, r_n)$ is lower bounded as

$$\frac{a_n}{2d^{1+1/p}} \leq a'_n. \tag{17}$$

First, we show that (17) holds under the Chebyshev distance. Let an RGG and a DGG be obtained by connecting two nodes if the Chebyshev distance between them is at most $r_n > 0$. Recall that the Chebyshev distance corresponds to the metric given by the ℓ_∞-norm. Then, the degree of a d-dimensional DGG with n nodes formed by using the Chebyshev distance is given by [16]

$$\begin{aligned} a'_n &= \left(2\lfloor n^{1/d} r_n \rfloor + 1\right)^d - 1 \\ &= \left(2\lfloor a_n^{1/d} \rfloor + 1\right)^d - 1, \end{aligned}$$

where $\lfloor x \rfloor$ is the integer part, i.e., the greatest integer less than or equal to x.

For $p = \infty$, we have

$$\begin{aligned} \frac{a_n}{2d} \leq a'_n \iff a_n \leq 2d a'_n \iff a_n &\leq 2d\left(2\lfloor a_n^{1/d} \rfloor + 1\right)^d - 2d \\ \iff (a_n + 2d) &\leq 2d\left(2\lfloor a_n^{1/d} \rfloor + 1\right)^d. \end{aligned}$$

Notice that $\lfloor a_n^{1/d} \rfloor \geq (a_n^{1/d} - 1)$, then it is sufficient to show that

$$\begin{aligned} (a_n + 2d) \leq 2d\left(2(a_n^{1/d} - 1) + 1\right)^d &\iff (a_n + 2d) \leq 2d\left(2a_n^{1/d} - 1\right)^d \\ &\iff \left(\frac{1}{2d} + \frac{1}{a_n}\right) \leq \left(2 - \frac{1}{a_n^{1/d}}\right)^d. \end{aligned}$$

By taking the log in both sides of the last inequality, yields

$$\ln\left(\frac{1}{2d} + \frac{1}{a_n}\right) \leq d \ln\left(2 - \frac{1}{a_n^{1/d}}\right).$$

Consequently, under the Chebyshev distance, (17) holds for $a_n \geq \frac{2d}{2d-1}$.

Next, we show that (17) holds under any ℓ_p-metric, $p \in [1,\infty]$. Let b_n and b'_n be the degrees of an RGG and a DGG formed by connecting each two nodes when $d^{1/p}\|x_i - x_j\|_\infty \leq r_n$. This simply means that the graphs are obtained using the Chebyshev distance with a radius equal to $\dfrac{r_n}{d^{1/p}}$. Then, the degree of the DGG can be written as

$$b'_n = \left(2\left\lfloor b_n^{1/d} \right\rfloor + 1\right)^d - 1.$$

When $p = \infty$, we have that (17) holds. Therefore, we deduce that for $b_n \geq \dfrac{2d}{2d-1}$, we get

$$\frac{b_n}{2d} \leq b'_n.$$

Note that for any ℓ_p-metric with $p \in [1,\infty)$ in $\mathbb{R}^d$, we have

$$\|x_i - x_j\|_p \leq d^{1/p}\|x_i - x_j\|_\infty.$$

Then the number of nodes a'_n in the DGG that falls in the ball of radius r_n is greater or equal than b'_n, i.e., $b'_n \leq a'_n$. Therefore,

$$\frac{b_n}{2d} = \frac{a_n}{2d^{1+1/p}} \leq b'_n \leq a'_n.$$

Hence, for $a_n \geq \frac{2d^{1+1/p}}{2d-1}$, we get

$$\frac{a_n}{2d^{1+1/p}} \leq a'_n.$$

□

E Proof of Lemma 4

In this appendix, we provide the eigenvalues of the regularized normalized Laplacian matrix for a DGG using the Chebyshev distance. Then, the degree of a vertex in $G(\mathcal{D}_n, r_n)$ is given as [16]

$$a'_n = (2k_n + 1)^d - 1, \quad \text{with } \; k_n = \lfloor \mathrm{N} r_n \rfloor \,,$$

where $\lfloor x \rfloor$ is the integer part, i.e., the greatest integer less than or equal to x. The regularized normalized Laplacian can be written as

$$\hat{\mathcal{L}}(\mathcal{D}_n) = \mathbf{I} - \frac{1}{(a'_n + \alpha)}\mathbf{A} - \frac{\alpha}{n(a'_n + \alpha)}\mathbf{1}\mathbf{1}^T, \tag{A}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{1}^T = [1, ..., 1]^T$ is the vector of all ones and $\mathbf{A}$ is the adjacency matrix defined as

$$\mathbf{A}_{ij} = \begin{cases} 1, \; \|x_i - x_j\|_p \leq r_n, \quad i \neq j, \; \text{ and } \; p \in [1, \infty], \\ 0, \; \text{otherwise.} \end{cases}$$

When $d = 1$, the adjacency matrix $\mathbf{A}$ of a DGG in $\mathbb{T}^1$ with n nodes is a circulant matrix. A well known result appearing in [28], states that the eigenvalues of a circulant matrix are given by the discrete Fourier transform (DFT) of the first row of the matrix. When $d > 1$, the adjacency matrix of a DGG is no longer circulant but it is block circulant with $\mathrm{N}^{d-1} \times \mathrm{N}^{d-1}$ circulant blocks, each of size $\mathrm{N} \times \mathrm{N}$. The author in [16], pages 85-87, utilizes the result in [28], and shows that the eigenvalues of the adjacency matrix in $\mathbb{T}^d$ are found by taking the d-dimensional DFT of an N^d tensor of rank d obtained from the first block row of (A)

$$\lambda_{m_1,...,m_d} = \sum_{h_1,...,h_d=0}^{\mathrm{N}-1} c_{h_1,...,h_d} \exp\left(-\frac{2\pi i}{\mathrm{N}}\mathbf{m}.\mathbf{h}\right), \tag{18}$$

where $\mathbf{m}$ and $\mathbf{h}$ are vectors of elements m_i and h_i, respectively, with $m_1, ..., m_d \in \{0, 1, ..., \mathrm{N}-1\}$ and $c_{h_1,...,h_d}$ defined as [16]

$$c_{h_1,...,h_d} = \begin{cases} 0, \text{ for } \; k_n < h_1, ..., h_d \leq \mathrm{N} - k_n - 1 \text{ or } h_1, ...h_d = 0, \\ 1, \text{ otherwise.} \end{cases} \tag{19}$$

The eigenvalues of the block circulant matrix $\mathbf{A}$ follow the spectral decomposition [16], page 86,

$$\mathbf{A} = \mathbf{F}^H \mathbf{\Lambda} \mathbf{F},$$

where $\mathbf{\Lambda}$ is a diagonal matrix whose entries are the eigenvalues of $\mathbf{A}$, and $\mathbf{F}$ is the d-dimensional DFT matrix. It is well known that when $d = 1$, the DFT of an $n \times n$ matrix is the matrix of the same size with entries

$$\mathbf{F}_{m,k} = \frac{1}{\sqrt{n}} \exp\left(-2\pi i m k/n\right) \quad \text{for } \; m, k = \{0, 1, ..., n-1\}.$$

When $d > 1$, the block circulant matrix $\mathbf{A}$ is diagonalized by the d-dimensional DFT matrix $\mathbf{F} = \mathbf{F}_{\mathrm{N}_1} \bigotimes ... \bigotimes \mathbf{F}_{\mathrm{N}_d}$, i.e., tensor product, where $\mathbf{F}_{\mathrm{N}_d}$ is the N_d-point DFT matrix.

Notice that all the matrices in (A) have a common eigenvector that is $\left(\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}}\right)$ and this eigenvector coincides with the first row and column of $\mathbf{F}$. Then, $\left(\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}}\right)$ is also an eigenvector of $\hat{\mathcal{L}}(\mathcal{D}_n)$.

The regularized normalized Laplacian can be expressed as

$$\begin{aligned}\hat{\mathcal{L}}(\mathcal{D}_n) &= \mathbf{I} - \frac{1}{(a'_n+\alpha)}\mathbf{F}^H\mathbf{\Lambda}\mathbf{F} - \frac{\alpha}{n(a'_n+\alpha)}\mathbf{1}\mathbf{1}^T \\ &= \mathbf{F}^H\left(\mathbf{I} - \frac{1}{(a'_n+\alpha)}\mathbf{\Lambda} - \frac{\alpha}{n(a'_n+\alpha)}\mathbf{F}\mathbf{1}\mathbf{1}^T\mathbf{F}^H\right)\mathbf{F} \\ &= \mathbf{F}^H\left(\mathbf{I} - \frac{1}{(a'_n+\alpha)}\mathbf{\Lambda} - \frac{n\alpha}{n(a'_n+\alpha)}\mathbf{e}_1^T\mathbf{e}_1\right)\mathbf{F} \\ &= \mathbf{F}^H\mathbf{\Lambda}_1\mathbf{F}, \end{aligned} \tag{20}$$

where $\mathbf{e}_1 = (1, 0...0)$ and $\mathbf{\Lambda}_1 = \left(\mathbf{I} - \frac{1}{(a'_n+\alpha)}\mathbf{\Lambda} - \frac{n\alpha}{n(a'_n+\alpha)}\mathbf{e}_1^T\mathbf{e}_1\right)$ is a diagonal matrix whose diagonal elements are the eigenvalues of $\hat{\mathcal{L}}(\mathcal{D}_n)$. Then, from (20), the derivation of the eignevalues of $\hat{\mathcal{L}}(\mathcal{D}_n)$ reduces to the derivation of the eigenvalues of the normalized adjacency matrix $\mathbf{A}'$.

The normalized adjacency matrix $\mathbf{A}'$ is defined as

$$\mathbf{A}' = \frac{1}{(a'_n+\alpha)}\mathbf{A}.$$

By using (18) and (19), the eigenvalues of $\mathbf{A}'$ for a DGG in $\mathbb{T}^d$ are given as

$$\begin{aligned}\lambda'_{m_1,\dots,m_d} &= \frac{1}{(a'_n+\alpha)}\left[\sum_{h_1,\dots,h_d=0}^{\mathrm{N}-1}\exp\left(-\frac{2\pi i\mathbf{mh}}{\mathrm{N}}\right) - \sum_{h_1,\dots,h_d=k_n+1}^{\mathrm{N}-k_n-1}\exp\left(-\frac{2\pi i\mathbf{mh}}{\mathrm{N}}\right)\right] - \frac{1}{(a'_n+\alpha)} \\ &= -\frac{1}{(a'_n+\alpha)}\sum_{h_1,\dots,h_d=k_n+1}^{\mathrm{N}-k_n-1}\exp\left(-\frac{2\pi i\mathbf{mh}}{\mathrm{N}}\right) - \frac{1}{(a'_n+\alpha)}\end{aligned}$$

$$\begin{aligned}\lambda'_{m_1,\dots,m_d} &= -\frac{1}{(a'_n+\alpha)}\prod_{s=1}^{d}\frac{\left(e^{\frac{-2im_s\pi}{\mathrm{N}}k_n} - e^{\frac{2im_s\pi}{\mathrm{N}}(1+k_n)}\right)}{\left(-1+e^{\frac{2im_s\pi}{\mathrm{N}}}\right)} - \frac{1}{(a'_n+\alpha)} \\ &= \frac{1}{(a'_n+\alpha)}\prod_{s=1}^{d}\frac{\left(e^{\frac{2im_s\pi}{\mathrm{N}}(1+k_n)} - e^{\frac{-2im_s\pi}{\mathrm{N}}k_n}\right)}{\left(-1+e^{\frac{2im_s\pi}{\mathrm{N}}}\right)} - \frac{1}{(a'_n+\alpha)} \\ &= \frac{1}{(a'_n+\alpha)}\prod_{s=1}^{d}\frac{\sin(\frac{m_s\pi}{\mathrm{N}}(2k_n+1))}{\sin(\frac{m_s\pi}{\mathrm{N}})} - \frac{1}{(a'_n+\alpha)}.\end{aligned}$$

Then, we conclude that the eigenvalues of $\hat{\mathcal{L}}(\mathcal{D}_n)$ for n finite are given by

$$\begin{aligned}\hat{\lambda}_{m_1,\dots,m_d} &= 1 - \frac{1}{(a'_n+\alpha)}\prod_{s=1}^{d}\frac{\sin(\frac{m_s\pi}{\mathrm{N}}(2k_n+1))}{\sin(\frac{m_s\pi}{\mathrm{N}})} + \frac{1-\alpha\delta_{m_1,\dots,m_d}}{(a'_n+\alpha)} \\ \hat{\lambda}_{m_1,\dots,m_d} &= 1 - \frac{1}{(a'_n+\alpha)}\prod_{s=1}^{d}\frac{\sin(\frac{m_s\pi}{\mathrm{N}}(a'_n+1)^{1/d})}{\sin(\frac{m_s\pi}{\mathrm{N}})} + \frac{1-\alpha\delta_{m_1,\dots,m_d}}{(a'_n+\alpha)},\end{aligned}$$

with $m_1, ..., m_d \in \{0, ...\mathrm{N}-1\}$ and $\delta_{m_1,\dots,m_d} = 1$ when $m_1, ..., m_d = 0$ otherwise $\delta_{m_1,\dots,m_d} = 0$.

In particular, as $\alpha \to 0$, the eigenvalues of $\mathcal{L}(\mathcal{D}_n)$ in the connectivity regime are given by

$$\lambda_{m_1,\dots,m_d} = 1 - \frac{1}{a'_n}\prod_{s=1}^{d}\frac{\sin(\frac{m_s\pi}{\mathrm{N}}(a'_n+1)^{1/d})}{\sin(\frac{m_s\pi}{\mathrm{N}})} + \frac{1}{a'_n}, \tag{21}$$

with $m_1, ..., m_d \in \{0, ..., \mathrm{N}-1\}$.

In the thermodynamic regime, for $s \in \{0, ..., d\}$ we let $f_s = \frac{m_s}{\mathrm{N}}$ then as $n \to \infty$, $f_s \in \mathbb{Q} \cap [0, 1]$ where $\mathbb{Q}$ denotes the set of rational numbers. Therefore, for $\gamma \geq 1$, the eigenvalues of $\hat{\mathcal{L}}(\mathcal{X}_n)$ can be approximated by the eigenvalues of $\hat{\mathcal{L}}(\mathcal{D}_n)$ given as

$$\hat{\lambda}_{f_1,...,f_d} = 1 - \frac{1}{(\gamma' + \alpha)} \prod_{s=1}^{d} \frac{\sin(\pi f_s (\gamma' + 1)^{1/d})}{\sin(\pi f_s)} + \frac{1 - \alpha \delta_{f_1,...,f_d}}{(\gamma' + \alpha)}, \tag{22}$$

where $\gamma' = (2 \lfloor \gamma^{1/d} \rfloor + 1)^d - 1$ and $\delta_{f_1,...,f_d} = 1$ when $f_1, ..., f_d = 0$, otherwise $\delta_{f_1,...,f_d} = 0$. □

References

1. Z. Bai and J. W. Silverstein, *Spectral analysis of large dimensional random matrices*. Springer, 2010.
2. Z. D. Bai, "Methodologies in spectral analysis of large dimensional random matrices, a review," *Statistica Sinica*, vol. 9, pp. 611–677, 1999.
3. I. J. Farkas, I. Derényi, A.-L. Barabási, and T. Vicsek, "Spectra of "real-world" graphs: Beyond the semicircle law," *Physical Review E*, vol. 64, no. 2, p. 026704, 2001.
4. P. Van Mieghem, *Graph spectra for complex networks*. Cambridge University Press, 2010.
5. R. Couillet and M. Debbah, *Random matrix methods for wireless communications*. Cambridge University Press, 2011.
6. P. Erdos, "On random graphs," *Publicationes mathematicae*, vol. 6, pp. 290–297, 1959.
7. Z. J. Haas, J. Deng, B. Liang, P. Papadimitratos, and S. Sajama, "Wireless ad hoc networks," *Encyclopedia of Telecommunications*, 2002.
8. J. Yick, B. Mukherjee, and D. Ghosal, "Wireless sensor network survey," *Computer networks*, vol. 52, no. 12, pp. 2292–2330, 2008.
9. V. M. Preciado and A. Jadbabaie, "Spectral analysis of virus spreading in random geometric networks," *IEEE Conference on Decision and Control*, 2009.
10. A. Ganesh, L. Massoulié, and D. Towsley, "The effect of network topology on the spread of epidemics," in *Proc. of IEEE Conference on Computer Communications (INFOCOM)*, 2005.
11. M. Penrose, *Random geometric graphs*. Oxford University Press, 2003.
12. C. Marshall, J. Cruickshank, and C. O'Riordan, "Social network analysis of clustering in random geometric graphs."
13. M. Maier, M. Hein, and U. von Luxburg, "Optimal construction of k-nearest-neighbor graphs for identifying noisy clusters," *Theoretical Computer Science*, vol. 410, no. 19, pp. 1749–1764, 2009.
14. A. M. Sadri, S. Hasan, S. V. Ukkusuri, and J. E. S. Lopez, "Analyzing social interaction networks from twitter for planned special events," *arXiv preprint arXiv:1704.02489*, 2017.
15. L. Page, S. Brin, R. Motwani, and T. Winograd, "The pagerank citation ranking: Bringing order to the web." Stanford InfoLab, Tech. Rep., 1999.
16. A. Nyberg, "The Laplacian spectra of random geometric graphs," Ph.D. dissertation, 2014.
17. A. Nyberg, T. Gross, and K. E. Bassler, "Mesoscopic structures and the Laplacian spectra of random geometric graphs," *Journal of Complex Networks*, vol. 3, no. 4, pp. 543–551, 2015.
18. C. P. Dettmann and G. Knight, "Symmetric motifs in random geometric graphs," *Journal of Complex Networks*, vol. 6, no. 1, pp. 95–105, 2017.
19. N. El Karoui, "The spectrum of kernel random matrices," *The Annals of Statistics*, vol. 38, no. 1, pp. 1–50, 2010.
20. T. Jiang, "Distributions of eigenvalues of large Euclidean matrices generated from lp balls and spheres," *Linear Algebra and its Applications*, vol. 473, pp. 14–36, 2015.
21. C. Bordenave, "Eigenvalues of Euclidean random matrices," *Random Structures & Algorithms*, vol. 33, no. 4, pp. 515–532, 2008.
22. P. Blackwell, M. Edmondson-Jones, and J. Jordan, *Spectra of adjacency matrices of random geometric graphs*. Unpublished, 2007.
23. S. Rai, "The spectrum of a random geometric graph is concentrated," *Journal of Theoretical Probability*, vol. 20, no. 2, pp. 119–132, 2007.
24. K. Avrachenkov, B. Ribeiro, and D. Towsley, "Improving random walk estimation accuracy with uniform restarts," in *International Workshop on Algorithms and Models for the Web-Graph*. Springer, 2010, pp. 98–109.

25. J. C. Taylor, *An introduction to measure and probability*. Springer Science & Business Media, 2012.
26. T. Müller, "Two-point concentration in random geometric graphs," *Combinatorica*, vol. 28, no. 5, p. 529, 2008.
27. S. Janson, T. Luczak, and A. Rucinski, *Random graphs*. John Wiley & Sons, 2011.
28. R. M. Gray, "Toeplitz and circulant matrices: A review," *Foundations and Trends® in Communications and Information Theory*, vol. 2, no. 3, pp. 155–239, 2006.