Spectral Analysis of Random Geometric Graphs

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Ph.D. Defense
May 29, France

Outline

What is a Network?

- Collection of connected objects

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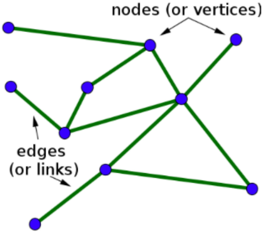
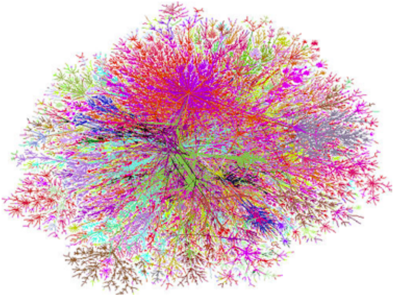
- Mathematically, objects are referred to as nodes or vertices and the connections are referred to as edges



Examples of Networks



Graphical representation of part of the Internet

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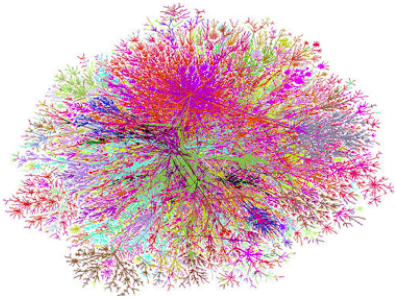

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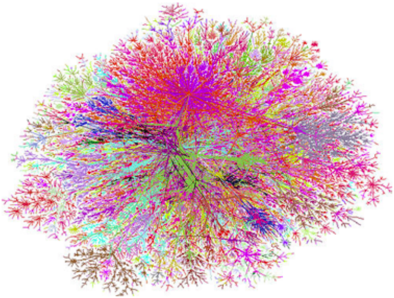

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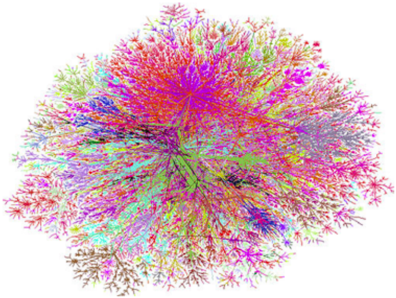

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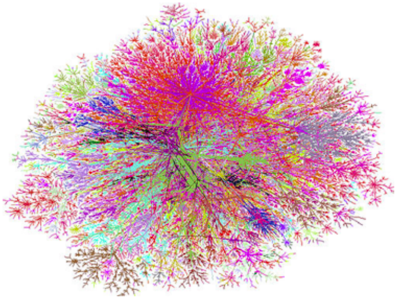

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  - Random geometric graph (RGG)

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RGG Application

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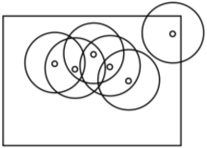

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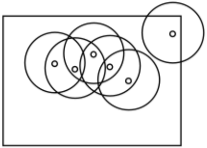

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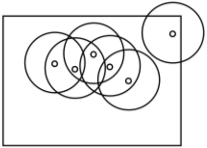

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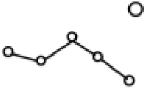

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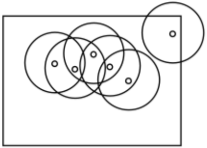

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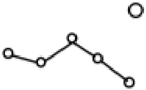

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- Ad-hoc network connectivity

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- Associate a matrix with a RGG
- Matrix eigenvalues $\Leftrightarrow$ Graph properties

RGG Matrix Representation

- Adjacency matrix $\mathbf{A}^{\mathcal{X}_n} \in \mathbb{R}^{n \times n}$

$$A_{ij}^{\mathcal{X}_n} = \begin{cases} 1, & \text{if } x_i \sim x_j \text{ and } x_i \neq x_j, \\ 0, & \text{otherwise.} \end{cases}$$

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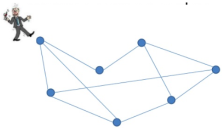
- Transition matrix $\mathbf{P}^{\mathcal{X}_n}$



• Random walk on a graph: stochastic process which randomly jumps from vertex to another vertex

$$\mathbf{P}^{\mathcal{X}_n} = \mathbf{D}^{-1}\mathbf{A}$$

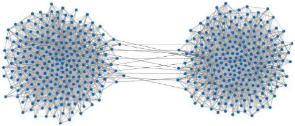
• d_i: degree of a vertex x_i

$$d_i = \sum_j A_{ij}^{\mathcal{X}_n}.$$

• $\mathbf{D} \in \mathbb{R}^{n\times n}$: diagonal matrix of vertex degrees

RGG Matrix Representation

- Spectrum of $\mathbf{P}^{\mathcal{X}_n}$ detect community structures of the network



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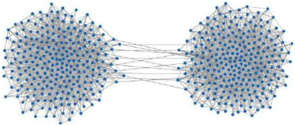

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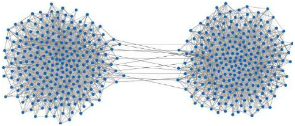

- $\mathbf{P}^{\mathcal{X}_n}$ is not symmetric

▶ Symmetric normalized Laplacian matrix $\boldsymbol{\mathcal{L}}^{\mathcal{X}_n} \in \mathbb{R}^{n \times n}$

$$\boldsymbol{\mathcal{L}}^{\mathcal{X}_n} = \mathbf{I}_n - \mathbf{D}^{-1/2}\mathbf{A}^{\mathcal{X}_n}\mathbf{D}^{-1/2}$$

- $\mathbf{I}_n \in \mathbb{R}^{n \times n}$: identity matrix
- If λ is an eigenvalue of $\mathbf{P}^{\mathcal{X}_n}$ then $1 - \lambda$ is an eigenvalue of $\boldsymbol{\mathcal{L}}^{\mathcal{X}_n}$

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  - Random walks on graphs: probability of hitting times[a] is governed by the spectrum of the normalized Laplacian matrix

  - Network epidemics: time evolution of the infected population is governed by the spectral radius of the adjacency matrix

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Main Contributions

Contributions

- Contribution 1: analysis of the spectrum of the normalized Laplacian matrix[a] [b]
- Contribution 2: analysis of the spectrum of the adjacency matrix[c]
- Contribution 3: determine the spectral dimension of RGGs, a generalized dimension for irregular structures[d]

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[a] **M. Hamidouche**, L. Cottatellucci, and K. Avrachenkov, "On the Normalized Laplacian Spectra of Random Geometric Graphs" *Submitted to Journal of theoretical probability*

[b] **M. Hamidouche**, L. Cottatellucci, and K. Avrachenkov, "Spectral Bounds of the Regularized Normalized Laplacian for Random Geometric Graphs." *4th Graph Signal Processing Workshop, 2019*

[c] **M. Hamidouche**, L. Cottatellucci, and K. Avrachenkov, "Spectral Analysis of the Adjacency Matrix of Random Geometric Graphs." *57th Annual Allerton Conference on Communication, Control, and Computing, 2019*

[d] K. Avrachenkov, L. Cottatellucci and **M. Hamidouche**, "Eigenvalues and Spectral Dimension of Random Geometric Graphs" *8th International Conference on Complex Networks and their Applications, 2019*

Outline

1. Motivation

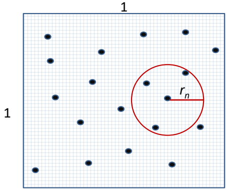
2. On the Normalized Laplacian Spectra of Random Geometric Graphs

3. Spectral Dimension (SD) of RGG

4. Summary and Perspectives

Random Geometric Graph

▶ Our focus: random geometric graphs (RGGs) $G(\mathcal{X}_n, r_n)$



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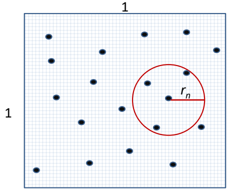

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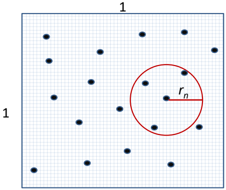

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- ℓ_p-metric:

$$\|x_i - x_j\|_p = \begin{cases} \left(\sum_{k=1}^{d} |x_i^{(k)} - x_j^{(k)}|^p\right)^{1/p} & p \in [1, \infty), \\ \max\{|x_i^{(k)} - x_j^{(k)}|,\ k \in [1, d]\} & p = \infty. \end{cases}$$

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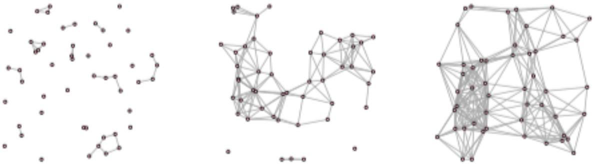
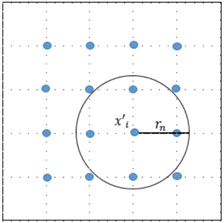
- Example: RGG in a unit Torus with $n = 50$ and r_n =0.1, 0.2, 0.3, respectively



Deterministic Geometric Graph (DGG) with Nodes in a Grid $G(\mathcal{D}_n, r_n)$



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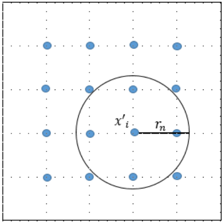

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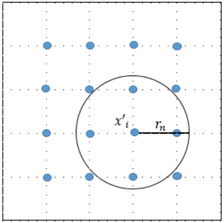

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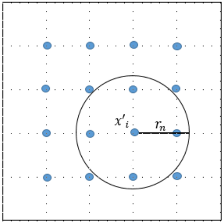

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Associated Matrices:

- $\mathbf{A}^{\mathcal{D}_n}$: adjacency matrix of $G(\mathcal{D}_n, r_n)$
- $\mathcal{L}^{\mathcal{D}_n}$: normalized Laplacian matrix of $G(\mathcal{D}_n, r_n)$

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▶ Contribution:

- Extend Rai work to the full range of the connectivity regime
- Analyze the spectrum in the thermodynamic regime
- Provide explicit expression for the eigenvalues of the normalized Laplacian matrix in the connectivity and thermodynamic regime

Hilbert-Schmidt Norm of the Difference Between $\mathcal{L}^{\mathcal{X}_n}$ and $\mathcal{L}^{\mathcal{D}_n}$

- Provide an upper bound on Hilbert-Schmidt norm of the difference between $\mathcal{L}^{\mathcal{X}_n}$ and $\mathcal{L}^{\mathcal{D}_n}$

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- Remark: In the thermodynamic regime, the error bound decreases for large γ

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Theorem 2: Eigenvalues of $\mathcal{L}^{\mathcal{D}_n}$

For $d \geq 1$, $\ell_\infty-$norm, the eigenvalues of $\mathcal{L}^{\mathcal{D}_n}$ are

$$\lambda_{m_1,\ldots,m_d} = 1 - \frac{1}{a_n} \prod_{s=1}^{d} \frac{\sin(\frac{m_s \pi}{\mathrm{N}}(a_n+1)^{1/d})}{\sin(\frac{m_s \pi}{\mathrm{N}})} + \frac{1}{a_n}$$

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Eigenvalues of the DGG Normalized Laplacian

- When $d = 1$, the adjacency matrix of a DGG is circulant and the eigenvalues are given by the discrete Fourier transform (DFT)

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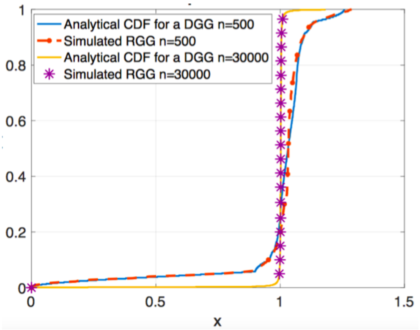
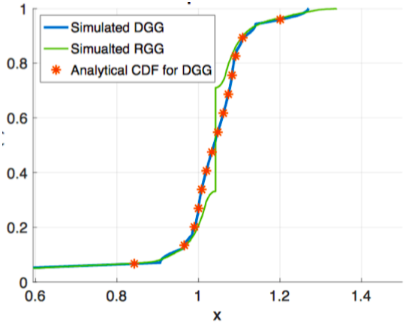
- When $a_n = \Omega(\log(n))$, $n \to \infty$, then all eigenvalues converge to 1

Numerical Results for the Connectivity Regime



- Curves corresponding to the RGG and the DGG match very well when n is large
- In the connectivity regime, as $n \to \infty$, the LSD of $\mathcal{L}(\mathcal{D}_n)$ converges to the Dirac measure at one

Numerical Results for the Thermodynamic Regime



- Empirical spectral distribution of an RGG and DGG with $d = 1$, $n = 30000$ vertices

- The gap that appears between the eigenvalue distributions of the RGG and the DGG is within the theoretical upper bound

Outline

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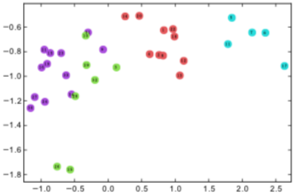

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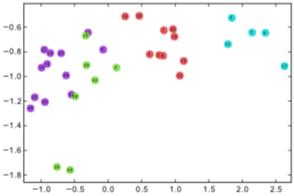

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Problem: Find an efficient method to estimate the dimension of networks modeled by random geometric graph

Spectral Dimension

Approach: Estimation of spectral dimension (SD)

- Spectral dimension: generalization of the Euclidean dimension of regular lattices to irregular structures such as graphs

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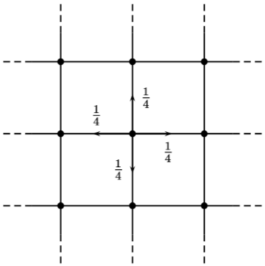
- When $\mathrm{P}_0(t) = t^{-\alpha}$ then $\alpha = \frac{d_s}{2}$

Random Walk on Regular Lattice



- Two-dimensional case: person walking randomly around a regular city

- On regular lattices, $\mathrm{P}_0(t)$ is controlled by the lattice (Euclidean) dimension d asymptotically

$$\mathrm{P}_0(t) \sim t^{-d/2} \quad \text{for} \quad t \to \infty$$

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- Long time limit of $\mathrm{P}_0(t)$ is linked to the behavior of $\rho(\lambda)$ for $\lambda \to 0$

Power-law tail Asymptotics

- $\rho(\lambda)$ follows a power-law tail asymptotics when $\lambda \to 0$

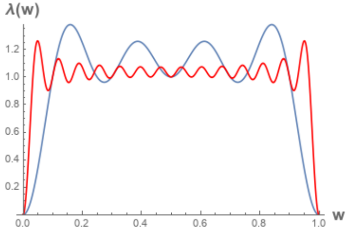
$$\rho(\lambda) \sim \lambda^{\gamma}, \quad \gamma > 0$$

- d_s can be described according to the asymptotic behavior of the normalized Laplacian empirical spectral distribution $F_n(x)$

$$\frac{d_s}{2} = \lim_{x \to 0} \frac{\log(F_n(x))}{\log(x)}$$

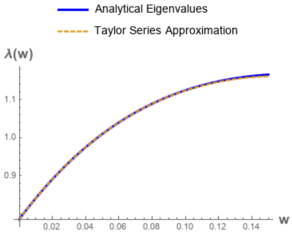
Eigenvalues Visualization in the Thermodynamic Regime

Analyze the behavior of the limiting spectral distribution (LSD) of the DGG regularized normalized Laplacian in a neighborhood of zero



- Eigenvalues of the DGG for $\gamma = 8$ (blue line) and $\gamma = 28$ (red line), $d = 1$
- DGG eigenvalues show a symmetry and the smallest ones are reached for small values of w

Spectral Dimension of RGG in the Thermodynamic Regime



- Approximation of the small eigenvalues by using Taylor series expansion of degree 2 around zero

$$\lambda(w) \approx \frac{\pi^2}{6(\gamma+\alpha)} w^{2/d} (\gamma' + 1)^{\frac{d+2}{d}}$$

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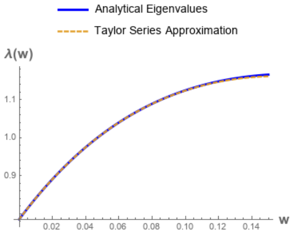

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Result

- The LSD of the DGG regularized normalized Laplacian in a neighborhood of zero follows a power-law asymptotics

$$F(x) \approx \frac{6^{d/2}(1+\gamma'+1)^{-\frac{2+d}{2}}}{\pi^d} x^{d/2}$$

- $d_s \sim d$

Recurrent and Transient RWs

- Random walk (RW) is recurrent if it visits its starting position infinitely often with probability one

- Random walk is transient if it visits its starting position finitely often with probability one

- If spectral dimension exists, then
  - RW is recurrent if $d_s \leq 2$
  - RW is transient if $d_s > 2$

Outline

1 Motivation

2 On the Normalized Laplacian Spectra of Random Geometric Graphs

3 Spectral Dimension (SD) of RGG

4 Summary and Perspectives

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- In connectivity regime, when $n \to \infty$, the LSD of the normalized Laplacian matrix of RGG converges with high probability to the Dirac distribution at one
- Approximation of the SD of RGGs in the thermodynamic regime by the Euclidean dimension d

Perspectives

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- Investigate RGG with real world properties by sampling nodes uniformly in the hyperbolic space
- Investigate a random graph for community detection called the geometric block model in both connectivity and thermodynamic regime

List of Publications

1) **M. Hamidouche**, L. Cottatellucci, and K. Avrachenkov, "On the Normalized Laplacian Spectra of Random Geometric Graphs" *Submitted to Journal of theoretical probability*

2) **M. Hamidouche**, L. Cottatellucci, and K. Avrachenkov, "Spectral Dimension and Clustering in Random Geometric Graphs" To be submitted to the Special Issue of the Journal Applied Network Science.

3) K. Avrachenkov, L. Cottatellucci and **M. Hamidouche**[1], "Eigenvalues and Spectral Dimension of Random Geometric Graphs" *8th International Conference on Complex Networks and their Applications, 2019*

4) **M. Hamidouche**, L. Cottatellucci, and K. Avrachenkov, "Spectral Analysis of the Adjacency Matrix of Random Geometric Graphs." *57th Annual Allerton Conference on Communication, Control, and Computing, 2019*

5) **M. Hamidouche**, L. Cottatellucci, and K. Avrachenkov, "Spectral Bounds of the Regularized Normalized Laplacian for Random Geometric Graphs." *4th Graph Signal Processing Workshop, 2019*

6) **M. Hamidouche**, E. Bastug, J. Park, L. Cottatellucci, and M. Debbah, "Downlink Performance of Dense Antenna Deployment: To Distribute or Doncentrate?" in *IEEE 28th Annual International Symposium on Personal, Indoor, and Mobile Radio Communications (PIMRC), 2017*

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Thanks for Your Attention.

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